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Formulation of the Nonabelian Tensor Square of a Bieberbach Group

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***Abstract:** A Bieberbach set can be categorized as a torsion free crystallographic set. Some properties can be explored from the set such as the property of nonabelian tensor square. The nonabelian tensor square is one type of the homological factors of sets. This paper focused on a Bieberbach set with $C_2 \times C_2$ as the point set of lowest dimension three. The purpose of this paper is to determine the generalization of the formula of the nonabelian tensor square of one Bieberbach set with point set $C_2 \times C_2$ of lowest dimension three which is denoted by $S_2(3)$ up to dimensionn. The polycyclic presentation, the abelianization of $S_2(3)$ and the central subgroup of the nonabelian tensor square of $S_2(3)$ are also presented.*

***Keywords:** Bieberbach set, Nonabelian Tensor Square, Homological Functor.*

I. INTRODUCTION

The Bieberbach sets are also known as a torsion free crystallographic sets. The Bieberbach set satisfies the short exact sequence

$$1 \longrightarrow L \xrightarrow{\varphi} G \xrightarrow{\phi} P \longrightarrow 1,$$

where this set is basically an extension of a free abelian set L of finite rank by a finite point set P such that $G/\varphi(L) \cong P$. In this paper, one Bieberbach set, which is the second set of lowest three dimension with $C_2 \times C_2$ as the group point is explored.

The nonabelian tensor square is an interesting property of crystallographic set that can be revealed. Let G be a set, the nonabelian tensor square of G, denoted by $G \otimes G$, is generated by generators $g_1 \otimes g_2$, for any $g_1, g_2 \in G$, satisfies the relations

$$g_1g_2 \otimes g_3 = \left({}^{g_1}g_2 \otimes {}^{g_1}g_3\right)\left(g_1 \otimes g_2\right) \text{ and}$$

$$g_1 \otimes g_3g_4 = \left({}^{g_3}g_1 \otimes {}^{g_3}g_4\right) \text{ for all } g_1, g_2, g_3, g_4 \in G,$$

where ${}^{g_1}g_2 = g_1g_2g_1^{-1}$ is a left conjugation. By calculating the nonabelian tensor square of G, the standard presentation of the nonabelian tensor square of any group G can be expressed.

The nonabelian tensor squares have been introduced by Brown and Loday [1] and is the specialization of the nonabelian tensor product in general. Past researchers have discovered the computations of the nonabelian tensor squares of many sets. For example, Sarminet. al [2], Kappe and Bacon [3] have computed thenonabelian tensor squares of 2-generator nilpotent of class two sets, Blyth, Morse and Moravec [4] have computed the nonabelian tensor squares of free nilpotent sets and Blyth and Morse [5] have created the formula for calculating nonabelian tensor squares of polycyclic sets .

Masri [6] has started the study of the nonabelian tensor squares of Bieberbach sets with certain cyclic spot group in 2009 and focused on Bieberbach sets with cyclic point set of order two. Next, the other studies which are related to the determination of the formula of the nonabelian tensor squares of other Bieberbach sets with different spot groups have also been done by other researchers such as the dihedral set ([7], [8]), the cyclic set of order three and five [9], the symmetric point set ([10, [11]) and the elementary abelian 2-set point group [12]. The abelian cases for the nonabelian tensor squares of Bieberbach sets can be found in [6] and these findings lead to the generalization of the formula of nonabelian tensor squares of the sets up to dimension n. Basedon the generalizations in [6], the homological functors of the Bieberbach set such as the schur multiplier, the nonabelian exterior squares can be computed up to dimension n in [9]. Since the nonabelian tensor square of $S_2(3)$ is found to be abelian in [12],this motivates us to study the generalization of the formula of the nonabelian tensor square of this set so that the homological functors of set $S_2(3)$ can be determined.

Therefore, the main objective of this paper is to the determine the generalization of the formula of the nonabelian tensor squareof one Bieberbach set of dimension 3 with $C_2 \times C_2$ as the point set, $S_2(3)$ up to dimension n.Firstly, the polycyclic presentation of $S_2(3)$is given as the following [12] :

$$S_2(3) = \left\langle a_0, a_1, l_1, l_2, l_3 \;\middle|\; \begin{array}{l} a_0^2 = l_2^{-1}l_3^{-1}, a_1^2 = l_1^{-1}, {}^{a_0}a_1 = a_1 l_1^{-1} l_2^{-1}, \\ {}^{a_0}l_1 = l_1^{-1}, {}^{a_0}l_2 = l_2, {}^{a_0}l_3 = l_3, \\ {}^{a_1}l_1 = l_1, {}^{a_1}l_2 = l_2^{-1}, {}^{a_1}l_3 = l_3, \\ {}^{l_1}l_2 = l_2, {}^{l_1}l_3 = l_3, {}^{l_2}l_3 = l_3 \end{array} \right\rangle \quad (1)$$

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II. PRELIMINARIES

Firstly, some structural results related to the computation will be provided in this section.A set $\nu(G)$ was introduced by Rocco [13] in 1991 and will be used in the computation of the nonabelian tensor square of the set.

Definition 1 [13]

Let $\langle G|R\rangle$ be a presentation of set *G* with relation *R* and let G^{φ} be an isomorphic copy of *G* via the mapping $\varphi: a \to a^{\varphi}$ for all $a \in G$. The group $\nu(G)$ is defined to be

$$\nu(G) = \langle G, G^{\varphi} \,|\, R, R^{\varphi}, {}^{x}[a,b^{\varphi}] = [{}^{x}a, ({}^{x}b)^{\varphi}] = {}^{x\varphi}[a,b^{\varphi}], \forall x,a,b \in G \rangle. \quad (1)$$

The collection of ν *(G)* is related to the nonabelian tensor square of set and can be shown as given in Theorem 1.

Theorem 1 ([13],[14])

Let σ be a mapping from $G \otimes G$ into $[G, G^{\varphi}] < \nu(G)$, for any group *G* and defined by $\sigma(a \otimes b) = [a, b^{\varphi}]$ for all *a*, *b* in *G*. Then, σ is an isomorphism.

Thus, based on Theorem 1, the commutator calculus can be used for the tensor computation within the subgroup of ν *(G)*, $[G,G^{\varphi}]$.

Next, the computational method for polycyclic set developed in [5] is used in order to determine the formula of the nonabelian tensor square of set $S_2(3)$. A list of commutator identities in ν *(G)* with left conjugation are given as in the following. Let *a*, *b* and *c* be elements of a group *G*. Then,

$$[ab, c] = {}^{a}[b,c] \cdot [a,c] \quad (2)$$

$$[a, bc] = [a,b] \cdot {}^{b}[a,c] \quad (3)$$

$$[a^{-1}, b] = [a^{-1}, [a,b]^{-1}] \cdot [a,b]^{-1} \quad (4)$$

$$[a, b^{-1}] = [b^{-1}, [a,b]^{-1}] \cdot [a,b]^{-1} \quad (5)$$

$$[a^{-1}, b^{-1}] = [a^{-1}, [b^{-1}, [a,b]]] \cdot [b^{-1}, [a,b]] \cdot [a^{-1}, [a,b]] \cdot [a,b] \quad (6)$$

$${}^{c}[a,b] = [{}^{c}a, {}^{c}b] \quad (7)$$

Definition 2

Let *G* be a group. The quotient of group *G* by its derived subgroup, G' is called as the abelianization of group *G*, denoted by $G^{ab} = G/G'$.

The next proposition shows the relationship between the structure of the abelianization of *G* and the central subgroup of the nonabelian tensor square of *G*, denoted by $\nabla(G)$.

Proposition 1 [15]

Assume that *G* be a group and G^{ab} is finitely generated. Let G^{ab} is the straight result of the cyclic groups $\langle a_i G' \rangle$,for $i = 1, \ldots, s$ and set $E(G)$ to be $\langle [a_i, a_j^{\varphi}] \,|\, i < j \rangle [G, G'^{\varphi}]$.

Then the following hold:

(i) $\nabla(G)$ is generated by

$\{[a_i, a_i^{\varphi}], [a_i, a_j^{\varphi}][a_j, a_i^{\varphi}] \,|\, 1 \le i < j \le s\}$;

(ii) $[G, G^{\varphi}] = \nabla(G) E(G)$.

Proposition 2 [16]

Assume that *X*, *Y* and *Z* are abelian groups. Then, the properties of the ordinary tensor product of any two abelian setsare given as in the following.

(i) $C_0 \otimes X \cong X$,

(ii) $C_0 \otimes C_0 \cong C_0$,

(iii) $C_i \otimes C_j \cong C_{\gcd(i,j)}$, for $i, j \in ¢$, and

(iv) $X \otimes (Y \times Z) = (X \otimes Y) \times (X \otimes Z)$

Where C_0 is a cyclic group of infinite order.

Proposition 3 [6]

Assume that *G* be any Bieberbach set of dimension *r* with *P* as the point group and lattice group *L*. Let $A = G \times F_s^{ab}$ where F_s^{ab} be a free abelian group of rank *S*. Then *A* is a Bieberbach group of dimension $r+s$ with point group *P*.

Theorem 2 [17]

Assume that *G* be a group. Let κ be a commutator mapping from $G \otimes G$ into G' and defined by $\kappa(a \otimes b) = [a,b]$, for all a, b in *G*. Here, the kernel of κ is in the centre of the nonabelian tensor square of *G*.

Note that $G \otimes G$ is an ordinary tensor square for abelian groups when *G* is abelian. The nonabelian tensor square of two abelian groups, *X* and *Y* is given as follows:

Proposition 4 [17]

Assume that *G* be a group such that $G = X \times Y$. Then,

$G \otimes G = (X \times Y) \otimes (X \times Y)$

$= (X \otimes X) \times (X^{ab} \otimes Y^{ab}) \times (B^{ab} \otimes Y^{ab}) \times (Y \otimes Y)$

Where $X^{ab} = X/X'$ and $Y^{ab} = Y/Y'$ are the abelianizations of *X* and *Y* respectively.

The following proposition gives the derivative subgroup, the abelianization, the central subgroup of the nonabelian tensor square and the nonabelian tensor square of group$S_2(3)$.

Proposition 5 [18]

For group $S_2(3)$, and $a_0, a_1, l_1, l_2 \in S_2(3)$,

(i) The derived subgroup $S_2(3)' = \langle l_1^{-2}, l_1^{-1} l_2 \rangle$

(ii) The abelianization

$S_2(3)^{ab} = \langle a_0 S_2(3)', a_1 S_2(3)' \rangle \cong C_0 \times C_4$.

Proposition 6 [18]

Let $a_0, a_1 \in S_2(3)$. Then, the central subgroup of the nonabelian tensor square of group $S_2(3)$, $\nabla S_2(3)$ is given as follows:

$\nabla(S_2(3)) =$

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$\left\langle [a_0, a_0^{\varphi}], [a_1, a_1^{\varphi}], [a_0, a_1^{\varphi}][a_1, a_0^{\varphi}] \right\rangle \cong C_4 \times C_8^2 \times C_0.$

Theorem 3 [12]

The nonabelian tensor square of group $S_2(3)$ is found to be abelian and given as follows:

III. RESULTS

The Family of Bieberbach Group S_2 of dimensionn

By Proposition 3, the family of Bieberbach group S_2 of dimension*n* can be defined as follows:

Definition 3

The group $S_2(n) = S_2(3) \times F_{n-3}^{ab}$ for $n \geq 3$

is a Bieberbach group with point group $C_2 \times C_2$ of dimension *n*, where F_s^{ab} is the free abelian group of rank *s*.

By constructing $S_2(n)$ and also by using the normal tensor creation of abelian groups, the formulation of the nonabelian tensor square of S_2 of dimension*n* can be determined as given in Theorem 4.

The calculation of the Formulation of the Nonabelian Tensor Square of $S_2(n)$

Next, Theorem 4 gives the main result of this paper which is the formulation of the nonabelian tensor square of Bieberbach group S_2 of dimension *n*.

Theorem 4

For the Bieberbach group of $S_2(n)$,

$S_2(n) \otimes S_2(n) \cong C_2 \times C_4{}^{2n-5} \times C_8 \times C_0{}^{n^2-4n+6}$ For $n \geq 4$.

Proof.

By Definition 3, $S_2(n) = S_2(3) \times F_{n-3}^{ab}$ for $n \geq 3$. Then by Proposition 4,

$$S_2(n) \otimes S_2(n) = (S_2(3) \otimes S_2(3)) \times (S_2(3) \times F_{n-3}^{ab}) \times (F_{n-3}^{ab} \otimes S_2(3)^{ab}) \times (F_{n-3}^{ab} \otimes F_{n-3}^{ab}).$$

By Theorem 3,

$$S_2(3) \otimes S_2(3) \cong C_2 \times C_4 \times C_8 \times C_0^3.$$

Then, by Proposition 5 (ii), we have $S_2(3)^{ab} \cong C_0 \times C_4$.

By using Proposition 2(i),

$$S_2(3)^{ab} \otimes F_{n-3}^{ab} \cong (C_4 \times C_0) \otimes C_0^{n-3}$$
$$= (C_4 \otimes C_0^{n-3}) \times (C_0 \otimes C_0^{n-3})$$
$$= C_4^{n-3} \times C_0^{n-3}.$$

And by symmetry,

$$F_{n-3}^{ab} \otimes S_2(3)^{ab} = C_4^{n-3} \times C_0^{n-3}.$$

Finally, by Proposition 2(ii) we have,

$$F_{n-3}^{ab} \otimes F_{n-3}^{ab} = C_0^{n-3} \times C_0^{n-3}$$
$$\cong C_0^{(n-3)^2}.$$

By collecting terms, then

$$S_2(n) \otimes S_2(n)$$
$$\cong C_2 \times C_4 \times C_8 \times C_0^3 \times C_4^{n-3} \times C_0^{n-3} \times C_4^{n-3} \times C_0^{n-3} \times C_0^{(n-3)^2}$$
$$= C_2 \times C_4^{1+(n-3)+(n-3)} \times C_8 \times C_0^{3+(n-3)+(n-3)+(n-3)^2}$$
$$= C_2 \times C_4^{2n-5} \times C_8 \times C_0^{n^2-4n+6}$$

Which completes the proof.

IV. CONCLUSIONS

The formulation of the nonabelian tensor square of one Bieberbach group is discussed in this paper, $S_2(3)$ up to dimension *n* is developed. This generalization can be used for by other researcher to compute and generalize the other types of homological functors of this group up to dimension *n*.

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