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SET-VALUED LEBESGUE AND RIESZ TYPE THEOREMS

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Abstract. In this paper, we continue a previous study concerning different types of pseudo-convergences of sequences of measurable functions with respect to set-valued non-additive monotonic set functions and we establish some pseudo-versions of Lebesgue and Riesz theorems in the set-valued case. We also characterize some important structural properties of fuzzy multimeasures.

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1. Introduction

As it is well-known, convergence theorems for sequences of measurable functions play a very important role in classical measure theory.

Relationships among different types of convergences such as almost everywhere convergence, almost uniform convergence and convergence in measure were especially described by the fundamental results contained in the Egoroff, Lebesgue and Riesz theorems (PRECUPANU [19]).

In non-additive measure theory, we mention the remarkable contributions of WANG and KLIR [30], PAP [18], DENNEBERG [1], LI and YASUDA [13], LI [9, 10], LI and LI [11], LI ET AL. [14], MUROFUSHI ET AL. [17], KAWABE [6, 7] concerning Egoroff's theorem, the papers of LI [9], SONG and LI [24] for Lebesgue's theorem or Sun [25] for Riesz's theorem and JIANG ET AL. [5] or TAKAHASHI ET AL. [27], HA ET AL. [3], LIU [15], LI ET AL. [12], LI [9], LI ET AL. [14], LI ET AL. [8], concerning different convergence theorems of sequences of measurable functions. We also remark the papers of MUROFUSHI [16], REN ET AL. [23], SUN [26], ZHANG [28], WANG [29] and many others.

Recently, motivated by the applied problems coming from mathematic economics, artificial intelligence, biomathematics and other important fields, some of the above mentioned results were generalized in the set-valued case. In this sense, we remark the paper of LIU [15], in which are given set-valued versions of Egoroff theorem and of Lebesgue theorem for sequences of set-valued measurable functions, our papers [20-22] concerning Egoroff and Lusin theorems for set-valued fuzzy multimeasures, or the paper of WU and LIU [31], which contains a set-valued version of Riesz theorem.

The aim of this paper is to continue for set-valued non-additive monotonic set functions, the investigation concerning different types of convergences and pseudo-convergences of sequences of measurable functions. Thus, we give several set-valued versions of Lebesgue theorem and a pseudo-version of Riesz's theorem, in which intercomes the property (PS) introduced by us in the set-valued case.

2. Terminology and notations

Let T be an abstract space, $\mathcal{A}$ a σ-algebra of subsets of T, X a real normed space with the origin 0, $\mathcal{P}_0(X)$ the family of all nonvoid subsets of X, $\mathcal{P}_f(X)$ the family of closed, nonvoid sets of X, $\mathcal{P}_{bf}(X)$ the family of all bounded, closed, nonvoid sets of X, $\mathcal{P}_{bfc}(X)$ the family of all bounded, closed, convex nonvoid sets of X and h the Hausdorff pseudometric on $\mathcal{P}_f(X)$ given by:

$$h(M,N) = \max\{e(M,N), e(N,M)\}, \text{ for every } M, N \in \mathcal{P}_f(X),$$

where $e(M,N) = \sup_{x\in M} d(x,N)$ is *the excess* of M over N.

It is known that $e(M,N) = 0$ if and only if $M \subset N$. Therefore, $e(M,N) = h(M,N)$, for every $M, N \in \mathcal{P}_f(X)$, with $N \subset M$. Also, $e(M,N) \le e(M,P) + e(P,N)$, for every $M, N, P \in \mathcal{P}_f(X)$. On $\mathcal{P}_{bf}(X)$, h becomes a metric [4].

We denote $|M| = h(M,\{0\})$, for every $M \in \mathcal{P}_f(X)$. We also denote $A \cap \mathcal{A} = \{E \subset A, E \in \mathcal{A}\}$, where A is a fixed set in $\mathcal{A}$.

Throughout the paper we shall use the following notions in the set valued case:

Definition 2.1 ([2], [20]-[22]). A set multifunction $\mu : \mathcal{A} \to \mathcal{P}_f(X)$ is said to be:

i) a *fuzzy multimeasure* if μ is monotone with respect to the inclusion of sets (i.e., $\mu(A) \subseteq \mu(B)$, for every $A, B \in \mathcal{A}$, with $A \subseteq B$) and $\mu(\emptyset) = \{0\}$.

ii) *continuous from below* if $\lim_{n\to\infty} h(\mu(A_n), \mu(A)) = 0$, for every increasing sequence of sets $(A_n)_n \subset \mathcal{A}$, with $A_n \nearrow A$.

iii) *continuous from above* if $\lim_{n\to\infty} h(\mu(A_n), \mu(A)) = 0$, for every decreasing sequence of sets $(A_n)_n \subset \mathcal{A}$, with $A_n \searrow A$.

iv) a *fuzzy multimeasure in the sense of Sugeno*, for short *(S)-fuzzy multimeasure,* if μ is a fuzzy multimeasure which is continuous from below and continuous from above.

v) *order continuous* if $\lim_{n\to\infty} |\mu(A_n)| = 0$, for every sequence of sets $(A_n)_n \subset \mathcal{A}$, with $A_n \searrow \emptyset$.

vi) *strongly order continuous* if $\lim_{n\to\infty} |\mu(A_n)| = 0$, for every sequence of sets $(A_n)_n \subset \mathcal{A}$, with $A_n \searrow A$ and $\mu(A) = \{0\}$.

vii) *pseudo-order continuous* if for every $B \in \mathcal{A}$ and every sequence of sets $(A_n)_n \subset \mathcal{A}$ with $A_n \subset B$, $n \in \mathbb{N}$ and $A_n \searrow A$, we have $\lim_{n\to\infty} |\mu(A_n)| = 0$ and $\mu(B \backslash A) = \mu(B)$.

viii) *null-additive* if $\mu(A \cup B) = \mu(B)$, for every disjoint $A, B \in \mathcal{A}$, with $\mu(A) = \{0\}$.

ix) *pseudo-null-additive* if $\mu(B \cup C) = \mu(C)$, whenever $A \in \mathcal{A}, B \in A \cap \mathcal{A}, C \in A \cap \mathcal{A}$ and $\mu(A \backslash B) = \mu(A)$.

x) a) *autocontinuous from below* (*autocontinuous from above*, respectively) if for every $A \in \mathcal{A}$ and every $(B_n)_n \subset \mathcal{A}$, with $\lim_{n\to\infty} |\mu(B_n)| = 0$, we have $\lim_{n\to\infty} h(\mu(A \backslash B_n), \mu(A)) = 0$ ($\lim_{n\to\infty} h(\mu(A \cup B_n), \mu(A)) = 0$, respectively).

b) *autocontinuous* if it is autocontinuous from above and autocontinuous from below.

xi) a) *pseudo-autocontinuous from above* (*pseudo-autocontinuous from below,* respectively) if for every $A \in \mathcal{A}$ and every $(B_n)_n \subset \mathcal{A}$, with $\lim_{n\to\infty} h(\mu(B_n \cap A), \mu(A)) = 0$, we have $\lim_{n\to\infty} h((\mu(A \backslash B_n) \cup C), \mu(C)) = 0$ (respectively, $\lim_{n\to\infty} h((\mu(B_n \cap C), \mu(C)) = 0)$, for every $C \in A \cap \mathcal{A}$.

b) *pseudo-autocontinuous* if it is pseudo-autocontinuous from above and pseudo-autocontinuous from below.

Definition 2.2. We say that a set multifunction $\mu : \mathcal{A} \to \mathcal{P}_f(X)$ *fulfils:*

i) [20, 22] *property* (S) if for any sequence of sets $(A_n)_n \subset \mathcal{A}$, with $\lim_{n\to\infty} |\mu(A_n)| = 0$, there exists a subsequence $(A_{n_k})_k$ of $(A_n)_n$ such that $\mu(\overline{\lim}_k A_{n_k}) = \{0\}$, where $\overline{\lim}_n E_n = \limsup_n E_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k$.

ii) [22] *property* (PS) if for any $A \in \mathcal{A}$ and any sequence of sets $(A_n)_n \subset A \cap \mathcal{A}$, with $\lim_{n\to\infty} h(\mu(A_n), \mu(A)) = 0$, there exists a subsequence $(A_{n_k})_k$ of $(A_n)_n$ such that $h(\mu(\underline{\lim}_k A_{n_k}), \mu(A)) = 0$, where $\underline{\lim}_n E_n = \liminf_n E_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} E_k$.

Unless stated otherwise, all over the paper we assume that $\mu : \mathcal{A} \to \mathcal{P}_f(X)$ is *a fuzzy* (*i.e., monotone*) *multimeasure.* By $\mathcal{M}$ we denote the class of all $\mathcal{A}$-measurable real-valued functions on $(T, \mathcal{A}, \mu)$, the space with the fuzzy multimeasure μ.

Definition 2.3. We consider arbitrary $\{f_n\} \subset \mathcal{M}$ and $f \in \mathcal{M}$. We say that:

i) [22] $\{f_n\}$ *converges μ-almost everywhere* (respectively, *pseudo-μ-almost everywhere*) *to f on A*, and denote it by $f_n \xrightarrow[A]{a.e.} f$ (respectively, $f_n \xrightarrow[A]{p.a.e.} f$) if there exists a subset $B \in A \cap \mathcal{A}$ such that $\mu(B) = \{0\}$ (respectively, $\mu(A \backslash B) = \mu(A)$) and $\{f_n\}$ is pointwise convergent to f on $A \backslash B$.

ii) [22] $\{f_n\}$ *converges in μ-measure* (respectively, *pseudo in μ-measure*) *to f on A*, and denote it by $f_n \xrightarrow[A]{\mu} f$ (respectively, $f_n \xrightarrow[A]{p.\mu} f$) if for every $\varepsilon > 0$, $\lim_{n\to\infty} |\mu(A_n(\varepsilon)| = 0$, where $A_n(\varepsilon) = \{t \in A; |f_n(t) - f(t)| \geq \varepsilon\}$ (respectively, $\lim_{n\to\infty} h(\mu(A \backslash A_n(\varepsilon)), \mu(A)) = 0$).

iii) [20, 22] $\{f_n\}$ *converges μ-almost uniformly* (respectively, *μ-pseudo-almost uniformly*) *to f on A* and denote it by $f_n \xrightarrow[A]{a.u.} f$ (respectively, $f_n \xrightarrow[A]{p.a.u.} f$) if there exists a decreasing sequence $\{A_k\}_{k\in\mathbb{N}} \subset A \cap \mathcal{A}$ such that $\lim_{k\to\infty} |\mu(A_k)| = 0$ (respectively, $\lim_{k\to\infty} h(\mu(A \backslash A_k), \mu(A)) = 0$) and for every fixed $k \in \mathbb{N}$, $\{f_n\}$ uniformly converges to f on $A \backslash A_k$ ($f_n \xrightarrow[A \backslash A_k]{u} f$).

Remark 2.4 ([22]). The following statements are equivalent:

a) μ is pseudo-null-additive;

b) $\mu(B \cap C) = \mu(C)$, whenever $A \in \mathcal{A}, B \in A \cap \mathcal{A}, C \in A \cap \mathcal{A}$ and $\mu(B) = \mu(A)$;

c) $\mu((A \backslash B) \cup C) = \mu(C)$, whenever $A \in \mathcal{A}, B \in A \cap \mathcal{A}, C \in A \cap \mathcal{A}$ and $\mu(B) = \mu(A)$.

3. Set-valued versions of Lebesgue theorems

In this section, we present some set-valued versions of Lebesgue theorem.

Firstly, using some ideas from [9] and [24], we establish several set-valued versions of Lebesgue theorem. In this way, we give some characterizations for several important asymptotic structural properties of monotone set multifunctions.

Theorem 3.1 (Lebesgue type). *Let be $A \in \mathcal{A}, f \in \mathcal{M}$ and $\{f_n\} \subset \mathcal{M}$. Then:*

i) $f_n \xrightarrow[A]{a.e.} f \Rightarrow f_n \xrightarrow[A]{\mu} f$ *if and only if μ is strongly order continuous;*

ii) $f_n \xrightarrow[A]{p.a.e.} f \Rightarrow f_n \xrightarrow[A]{p.\mu} f$ *if and only if μ is continuous from below;*

iii) $f_n \xrightarrow[A]{p.a.e.} f \Rightarrow f_n \xrightarrow[A]{\mu} f$ *if and only if μ is pseudo-order continuous;*

iv) If $\mu : \mathcal{A} \to \mathcal{P}_{bf}(X)$, then $f_n \xrightarrow[A]{a.e.} f \Rightarrow f_n \xrightarrow[A]{p.\mu} f$ *if and only if μ is null-additive and continuous from below.*

Proof. Take arbitrary $A \in \mathcal{A}$ and $f \in \mathcal{M}, \{f_n\} \subset \mathcal{M}$. We observe that the set C of points $t \in A$ at which $\{f_n\}$ is pointwise convergent to f can be written as $C = \bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcap_{i=n}^{\infty} (A \backslash B_i(\frac{1}{m}))$, where $B_i(\frac{1}{m}) = \{t \in A; |f_i(t) - f(t)| \geq \frac{1}{m}\}$, for every $m, i \in \mathbb{N}^*$. For every $m, n \in \mathbb{N}^*$, we denote $A_n^{(m)} = \bigcup_{i=n}^{\infty} B_i(\frac{1}{m})$ and $A^{(m)} = \bigcap_{n=1}^{\infty} A_n^{(m)}$. Then $A \backslash A^{(m)} = \bigcup_{n=1}^{\infty} (A \backslash A_n^{(m)}) = \bigcup_{n=1}^{\infty} \bigcap_{i=n}^{\infty} (A \backslash B_i(\frac{1}{m}))$ and $C = \bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} (A \backslash A_n^{(m)}) = \bigcap_{m=1}^{\infty} (A \backslash A^{(m)})$. We observe that for every fixed $m \in \mathbb{N}^*$, $A_n^{(m)} \underset{n\to\infty}{\searrow} A^{(m)}$

and so, $A \backslash A_n^{(m)} \underset{n\to\infty}{\nearrow} A \backslash A^{(m)}$. If there exists a set $B \in A \cap \mathcal{A}$ and $\{f_n\}$ is pointwise convergent to f on $A \backslash B$, then for every $m \in \mathbb{N}^*$,

$$A \backslash B \subset C \subset \bigcup_{n=1}^{\infty} \bigcap_{i=n}^{\infty} \left(A \backslash B_i \left(\frac{1}{m} \right) \right) = A \backslash A^{(m)} \subset A. \tag{1}$$

We also observe that

$$B_n \left(\frac{1}{m} \right) \subset A_n^{(m)}, \text{ for every } m, n \in \mathbb{N}^*. \tag{2}$$

i) *Necessity.* To prove that μ is strongly order continuous, let us consider $(A_n)_{n\in\mathbb{N}} \subset A \cap \mathcal{A}$, with $A_n \searrow \widetilde{A}$ and $\mu(\widetilde{A}) = \{0\}$. Then $\widetilde{A} \subset A_n \subset A$, for every $n \in \mathbb{N}$. We shall prove that $\lim_{n\to\infty} |\mu(A_n)| = 0$. For every $n \in \mathbb{N}$, we define the following functions:

$$f_n(t) = \begin{cases} 0, & \text{if } t \in A_n \\ 1, & \text{if } t \in A \backslash A_n \end{cases}$$

and

$$f(t) = \begin{cases} 0, & \text{if } t \in \widetilde{A} \\ 1, & \text{if } t \in A \backslash \widetilde{A}. \end{cases}$$

We observe that $f \in \mathcal{M}, \{f_n\} \subset \mathcal{M}$ and $\{f_n\}$ is pointwise convergent to 1 on $A \backslash \widetilde{A}$, so $f_n \xrightarrow[A]{a.e.} 1$. By virtue of hypothesis, $f_n \xrightarrow[A]{\mu} f$, whence $\lim_{n\to\infty} |\mu(\{t \in A; |f_n(t) - 1| \geq \frac{1}{2}\})| = 0$, which implies that $\lim_{n\to\infty} |\mu(A_n)| = 0$. This means that μ is strongly order continuous.

Sufficiency. Suppose μ is strongly order continuous and $f_n \xrightarrow[A]{a.e.} f$. Then there exists $B \in A \cap \mathcal{A}$ such $\mu(B) = \{0\}$ and $\{f_n\}$ is pointwise convergent to f on $A \backslash B$. By (1), for every $m \in \mathbb{N}^*, A^{(m)} \subset B$ and so, because μ is a fuzzy multimeasure, we get that $\mu(A^{(m)}) = \{0\}$.

Since for every fixed $m \in \mathbb{N}^*$, $A_n^{(m)} \underset{n\to\infty}{\searrow} A^{(m)}$ and μ is strongly order continuous, then for every $m \in \mathbb{N}^*$, $\lim_{n\to\infty} |\mu(A_n^{(m)})| = 0$. By (2), we get that for every $m \in \mathbb{N}^*$, $\lim_{n\to\infty} |\mu(B_n(\frac{1}{m}))| = 0$, that is, $f_n \xrightarrow[A]{\mu} f$.

ii) *Necessity*: To prove that μ is continuous from below, let us consider $(A_n)_{n\in\mathbb{N}} \subset A \cap \mathcal{A}$, with $A_n \nearrow \widetilde{A}$. Then $A_n \subset \widetilde{A} \subset A$, for every $n \in \mathbb{N}$.

We shall prove that $\lim\limits_{n\to\infty} h(\mu(A_n), \mu(\widetilde{A})) = 0$. For every $n \in \mathbb{N}$, we define $f, \{f_n\}$ as follows:

$$f_n(t) = \begin{cases} 0, & \text{if } t \in A \backslash A_n \\ 1, & \text{if } t \in A_n \end{cases}$$

and

$$f(t) = \begin{cases} 0, & \text{if } t \in A \backslash \widetilde{A} \\ 1, & \text{if } t \in \widetilde{A}. \end{cases}$$

We observe that $f \in \mathcal{M}, \{f_n\} \subset \mathcal{M}$ and $\{f_n\}$ is pointwise convergent to f on $\widetilde{A}$, so $f_n \xrightarrow[\widetilde{A}]{p.a.e.} f$.

Consequently, $f_n \xrightarrow[\widetilde{A}]{p.\mu} f$, whence $\lim_{n\to\infty} h(\mu(\widetilde{A}\backslash\{t \in \widetilde{A}; |f_n(t) - f(t)| \geq \frac{1}{2}\}), \mu(\widetilde{A})) = 0$ and so, $\lim_{n\to\infty} h(\mu(\widetilde{A}\backslash(A\backslash A_n)), \mu(\widetilde{A})) = 0$, which implies that $\lim_{n\to\infty} h(\mu(A_n), \mu(\widetilde{A})) = 0$, that is, μ is continuous from below.

Sufficiency: Suppose that μ is continuous from below and $f_n \xrightarrow[A]{p.a.e.} f$. Then there exists $B \in A \cap \mathcal{A}$ such that $\mu(A\backslash B) = \mu(A)$ and $\{f_n\}$ is pointwise convergent to f on $A\backslash B$. Because $\mu(A\backslash B) = \mu(A)$ and μ is a fuzzy multimeasure, by (1) we have

$$\mu(A\backslash B) = \mu(C) = \mu(\bigcup_{n=1}^{\infty}\bigcap_{i=n}^{\infty}(A\backslash B_i(\frac{1}{m}))) = \mu(A) = \mu(A\backslash A^{(m)}). \tag{3}$$

By (2), we get $\mu(A\backslash A_n^{(m)}) \subset \mu(A\backslash B_n(\frac{1}{m})) \subset \mu(A)$. Since μ is continuous from below, then for every $m \in \mathbb{N}$, $\lim_{n\to\infty} h(\mu(A\backslash A_n^{(m)}), \mu(A\backslash A^{(m)})) = 0$, that is, by (3), $\lim_{n\to\infty} h(\mu(A\backslash A_n^{(m)}), \mu(A)) = 0$.

Using Lemma 2.1 from [22], we have $\lim_{n\to\infty} h(\mu(A\backslash B_n(\frac{1}{m}), \mu(A)) = 0$, which says that $f_n \xrightarrow[A]{p.\mu} f$.

iii) *Necessity.* To prove that μ is pseudo order continuous, let us consider arbitrary $(A_n)_n \subset \mathcal{A}$ and $B \in \mathcal{A}$, with $A_n \subset B$, for every n, $A_n \searrow \widetilde{A} \in \mathcal{A}$ and $\mu(B\backslash\widetilde{A}) = \mu(B)$. We shall prove that $\lim_{n\to\infty} |\mu(A_n)| = 0$. For every $n \in \mathbb{N}$, we define $f, \{f_n\}$ as follows:

$$f_n(t) = \begin{cases} 0, & \text{if } t \in A_n \\ 1, & \text{if } t \in B\backslash A_n \end{cases}$$

and

$$f(t) = \begin{cases} 0, & \text{if } t \in \widetilde{A} \\ 1, & \text{if } t \in B \backslash \widetilde{A}. \end{cases}$$

We observe that $f \in \mathcal{M}, \{f_n\} \subset \mathcal{M}$ and $\{f_n\}$ is pointwise convergent to 1 on $B\backslash\widetilde{A}$, so $f_n \xrightarrow[B]{p.a.e.} 1$.

Consequently, $f_n \xrightarrow[B]{\mu} 1$, whence $\lim_{n\to\infty} |\mu(\{t \in B; |f_n(t) - 1| \geq \frac{1}{2}\})| = 0$, which implies $\lim_{n\to\infty} |\mu(A_n)| = 0$, that is, μ is pseudo-order continuous.

Sufficiency. Suppose μ is pseudo-order continuous and $f_n \xrightarrow[A]{p.a.e.} f$. There exists $B \in A \cap \mathcal{A}$ such that $\mu(A\backslash B) = \mu(A)$ and $\{f_n\}$ is pointwise convergent to f on $A\backslash B$. By (1), we have for every $m \in \mathbb{N}^*$, $\mu(A) = \mu(A\backslash A^{(m)})$. Consequently, because for every fixed $m \in \mathbb{N}^*$, $A_n^{(m)} \underset{n\to\infty}{\searrow} A^{(m)}$ and μ is pseudo-order continuous, then $\lim_{n\to\infty} |\mu(A_n^{(m)})| = 0$. By (2), we get that for every $m \in \mathbb{N}^*$, $\lim_{n\to\infty} |\mu(B_n(\frac{1}{m}))| = 0$, that is, $f_n \xrightarrow[A]{\mu} f$.

iv) *Necessity*: First, we prove that μ is continuous from below.

Let us consider the sequence $\{f_n\}$ from ii), which is pointwise convergent to f on $\widetilde{A}$, so $f_n \xrightarrow[\widetilde{A}]{a.e.} f$. By virtue of the hypothesis, $f_n \xrightarrow[\widetilde{A}]{p.\mu} f$, whence, as in ii), μ is continuous from below.

It only remains to prove that μ is null-additive. For this, take arbitrary disjoint $B_1, B_2 \in \mathcal{A}$, with $\mu(B_1) = \{0\}$. For every $n \in \mathbb{N}$, we define

$$f_n(t) = \begin{cases} 0, & \text{if } t \in B_1 \\ 1, & \text{if } t \in B_2 \end{cases}.$$

Then $f_n \xrightarrow[B_1\cup B_2]{a.e.} 1$, whence $f_n \xrightarrow[B_1\cup B_2]{p.\mu} 1$, so, $h(\mu((B_1\cup B_2)\backslash\{t \in B_1\cup B_2; |f_n(t) - 1| \geq \frac{1}{2}\})), \mu(B_1\cup B_2)) = 0$, which implies $h(\mu(B_2), \mu(B_1\cup B_2)) = 0$. Because $\mu : \mathcal{A} \to \mathcal{P}_{bf}(X)$, we finally have $\mu(B_1 \cup B_2) = \mu(B_2)$, so, μ is null-additive.

Sufficiency: Suppose μ is null-additive and continuous from below and $f_n \xrightarrow[A]{a.e.} f$. Since μ is null-additive, we have by Proposition 3.1 from [22] that $f_n \xrightarrow[A]{p.a.e.} f$ and by ii) we obtain that $f_n \xrightarrow[A]{p.\mu} f$. $\square$

4. Set-valued versions of Riesz theorem

In this section, we present some characterizations of pseudo-autocontinuity and we establish a pseudo-version of Riesz theorem.

Firstly, we mention the following generalization of the set-valued version of Riesz's theorem [31] and some of its consequences.

Theorem 4.1 (Riesz type theorem). *μ has property (S) if and only if for every $A \in \mathcal{A}$ and every $f \in \mathcal{M}, \{f_n\} \subset \mathcal{M}$, with $f_n \xrightarrow[A]{\mu} f$, there exists a subsequence $\{f_{n_k}\}_k$ of $\{f_n\}$ so that $f_{n_k} \xrightarrow[A]{a.e.} f$.*

By Theorem 3.1 i) and Theorem 4.1, we get:

Corollary 4.2. *Let be μ a strongly order continuous fuzzy multimeasure which fulfils property (S) and $A \in \mathcal{A}$, $f \in \mathcal{M}, \{f_n\} \subset \mathcal{M}$. Then:*

i) $f_n \xrightarrow[A]{a.e.} f$ implies $f_n \xrightarrow[A]{\mu} f$.

ii) If $f_n \xrightarrow[A]{\mu} f$, there is a subsequence $\{f_{n_k}\}_k$ of $\{f_n\}$ so that $f_{n_k} \xrightarrow[A]{a.e.} f$.

Since any continuous from above fuzzy multimeasure is strongly order continuous, by [20] (Corollary 4.7), and by the above considerations we get:

Remark 4.3. Let be arbitrary $A \in \mathcal{A}$, $f \in \mathcal{M}$ and $\{f_n\} \subset \mathcal{M}$ and suppose μ is continuous from above, with property (S). Then:

i) $f_n \xrightarrow[A]{a.e.} f \Leftrightarrow f_n \xrightarrow[A]{a.u.} f$;

ii) If $f_n \xrightarrow[A]{a.e.} f$, then $f_n \xrightarrow[A]{\mu} f$;

iii) If $f_n \xrightarrow[A]{\mu} f$, there exists a subsequence $\{f_{n_k}\}_k$ of $\{f_n\}$ such that $f_{n_k} \xrightarrow[A]{a.e.} f$.

Using some ideas from [25, 26], we also prove the following characterizations for pseudo-autocontinuity:

Theorem 4.4. *If $\mu : \mathcal{A} \to \mathcal{P}_{bf}(X)$ is a (S)-fuzzy multimeasure, then the following statements are equivalent:*

i) μ is pseudo-autocontinuous;

ii) μ is pseudo-autocontinuous from below;

iii) μ is pseudo-autocontinuous from above;

iv) μ *is pseudo-null-additive and fulfils property* (PS)*;*

v) μ *is pseudo-null-additive and for every* $A \in \mathcal{A}$ *and every* $(A_n)_n \subset \mathcal{A} \cap A$*, with* $\lim_{n\to\infty} h(\mu(A_n), \mu(A)) = 0$*, there exists a subsequence* $(A_{n_k})_k$ *so that* $\mu(A \backslash \underline{\lim}_k A_{n_k}) = \{0\}$.

Proof. In order to prove that i) $\Leftrightarrow$ ii) $\Leftrightarrow$ iii), it is sufficient to prove that ii) $\Leftrightarrow$ iii):

iii) $\Rightarrow$ ii): Let us assume that μ is pseudo-autocontinuous from above, but μ is not pseudo-autocontinuous from below. Then there exist $\varepsilon_0 > 0$, $A \in \mathcal{A}, C \in A \cap \mathcal{A}, (B_n) \subset A \cap \mathcal{A}$ so that $\lim_{n\to\infty} h(\mu(B_n), \mu(A)) = 0$, but $h(\mu(B_n \cap C), \mu(C)) = e(\mu(C), \mu(B_n \cap C)) > \varepsilon_0$, for every $n \in \mathbb{N}$. Let $n_1 = 1$. Since μ is pseudo-autocontinuous from above, then $\lim_{n\to\infty} h(\mu((B_{n_1} \cap C) \cup (A \backslash B_n)), \mu(B_{n_1} \cap C)) = 0$. Since for every $n \in \mathbb{N}$,

$$\varepsilon_0 < e(\mu(C), \mu(B_{n_1} \cap C)) \leq e(\mu(C), \mu((B_{n_1} \cap C) \cup (A \backslash B_n))) + e(\mu((B_{n_1} \cap C) \cup (A \backslash B_n)), \mu(B_{n_1} \cap C)),$$

then $\lim_{n\to\infty} e(\mu(C), \mu((B_{n_1} \cap C) \cup (A \backslash B_n))) > \varepsilon_0$, so, there exists $n_2 \in \mathbb{N}$ such that $n_2 > n_1$ and $e(\mu(C), \mu((B_{n_1} \cap C) \cup (A \backslash B_{n_2})) > \varepsilon_0$. Taking again into account that μ is pseudo-autocontinuous from above, then $\lim_{n\to\infty} h(\mu((B_{n_2} \cap C) \cup (A \backslash B_n)), \mu(B_{n_2} \cap C)) = 0$ and $\lim_{n\to\infty} h(\mu((B_{n_1} \cap C) \cup (A \backslash B_{n_2}) \cup (A \backslash B_n)), \mu((B_{n_1} \cap C) \cup (A \backslash B_{n_2}))) = 0$, so, as before, there exists $n_3 \in \mathbb{N}$ such that $n_3 > n_2$, $e(\mu(C), \mu((B_{n_2} \cap C) \cup (A \backslash B_{n_3}))) > \varepsilon_0$ and $e(\mu(C), \mu((B_{n_1} \cap C) \cup (A \backslash B_{n_2}) \cup (A \backslash B_{n_3}))) > \varepsilon_0$.

Analogously, there exists $n_4 \in \mathbb{N}$ such that $n_4 > n_3$, $e(\mu(C), \mu((B_{n_3} \cap C) \cup (A \backslash B_{n_4}))) > \varepsilon_0$, $e(\mu(C), \mu((B_{n_2} \cap C) \cup (A \backslash B_{n_3}) \cup (A \backslash B_{n_4}))) > \varepsilon_0$ and $e(\mu(C), \mu((B_{n_1} \cap C) \cup (A \backslash B_{n_2}) \cup (A \backslash B_{n_3}) \cup (A \backslash B_{n_4}))) > \varepsilon_0$. Recurrently, there exists $(B_{n_k})_k \subset (B_n)$ so that for every $k \in \mathbb{N}$,

$$e(\mu(C), \mu((B_{n_k} \cap C) \cup (A \backslash \bigcap_{t=k+1}^{\infty} B_{n_t}))) > \varepsilon_0.$$

Denote $D_k = B_{n_k} \cup (A \backslash \bigcap_{t=k+1}^{\infty} B_{n_t})$, for every $k \in \mathbb{N}$.

We observe that $\liminf_k D_k = \limsup_k D_k = A$ and, for every $k \in \mathbb{N}$, we have $\mu(C \cap (\bigcap_{t=k}^{\infty} D_t)) \subset \mu((B_{n_k} \cap C) \cup (A \backslash \bigcap_{t=k+1}^{\infty} B_{n_t}))$, so,

$$h(\mu(C \cap (\bigcap_{t=k}^{\infty} D_t)), \mu(C)) = e(\mu(C), \mu(C \cap (\bigcap_{t=k}^{\infty} D_t)))$$

$$= e(\mu(C), \mu(C \cap (\bigcap_{t=k}^{\infty} D_t)))$$

$$+ e(\mu(C \cap (\bigcap_{t=k}^{\infty} D_t)), \mu((B_{n_k} \cap C) \cup (A \setminus \bigcap_{t=k+1}^{\infty} B_{n_t})))$$

$$\geq e(\mu(C), \mu((B_{n_k} \cap C) \cup (A \setminus \bigcap_{t=k+1}^{\infty} B_{n_t}))) > \varepsilon_0.$$

Since $C \cap (\bigcap_{t=k}^{\infty} D_t) \underset{k\to\infty}{\nearrow} C \cap A = C$ and μ is continuous from below, then $\lim_{k\to\infty} h(\mu(C \cap (\bigcap_{t=k}^{\infty} D_t)), \mu(C)) = 0$, whence $0 \geq \varepsilon_0$, which is a contradiction.

ii) $\Rightarrow$ iv) a) Firstly, we prove that μ is pseudo-null-additive.

Indeed, for every $A \in \mathcal{A}, B \in A \cap \mathcal{A}, C \in A \cap \mathcal{A}$, with $\mu(A \setminus B) = \mu(A)$, applying the pseudo-autocontinuity from below of μ for $B_n = A \setminus B$, for every $n \in \mathbb{N}$, we get that $h(\mu((A \setminus B) \cap C), \mu(C)) = 0$. Because $\mu : \mathcal{A} \to \mathcal{P}_{bf}(X)$, then $\mu(C) = \mu(C \setminus B)$. Replacing C by $C \cup B$, we get $\mu(C \cup B) = \mu((C \cup B) \setminus B) = \mu(C \setminus B) = \mu(C)$, that is, μ is pseudo-null-additive.

b) Now, we prove that μ has PS. Suppose $A \in \mathcal{A}$, $(A_n) \subset A \cap \mathcal{A}$ are so that $\lim_{n\to\infty} h(\mu(A_n), \mu(A)) = 0$ and let be arbitrary $\varepsilon > 0$. Since μ is pseudo-autocontinuous from below, we get that $\lim_{n\to\infty} h(\mu(A_n \cap C), \mu(C)) = 0$, for every $C \in A \cap \mathcal{A}$. Since $\lim_{n\to\infty} h(\mu(A_n), \mu(A)) = 0$, there is $n_1 \in \mathbb{N}$ so that $h(\mu(A_{n_1}), \mu(A)) < \frac{\varepsilon}{2} < \varepsilon$. Because $\lim_{n\to\infty} h(\mu(A_n \cap A_{n_1}), \mu(A_{n_1})) = 0$, there is $n_2 \in \mathbb{N}$, so that $h(\mu(A_{n_2} \cap A_{n_1}), \mu(A_{n_1})) < \frac{\varepsilon}{2^2}$, so $h(\mu(A_{n_2} \cap A_{n_1}), \mu(A)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2^2} < \varepsilon$. Analogously, also taking into account that μ is continuous from above, we get that there exists $(A_{n_k})_k \subset (A_n)$ so that $h(\mu(\bigcap_{k=1}^{\infty} A_{n_k}), \mu(A)) \leq \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} = \varepsilon$.

Consequently, $e(\mu(A), \mu(\bigcap_{k=1}^{\infty} A_{n_k})) \leq \varepsilon$, whence, for every $s \in \mathbb{N}^*$, $e(\mu(A), \mu(\bigcap_{k=s}^{\infty} A_{n_k})) \leq e(\mu(A), \mu(\bigcap_{k=1}^{\infty} A_{n_k})) \leq \varepsilon$.

Since $\bigcap_{k=s}^{\infty} A_{n_k} \underset{s\to\infty}{\nearrow} \bigcup_{s=1}^{\infty} \bigcap_{k=s}^{\infty} A_{n_k}$ and μ is continuous from below, then $e(\mu(A), \mu(\bigcup_{s=1}^{\infty} \bigcap_{k=s}^{\infty} A_{n_k})) = 0$. Consequently, because $\mu : \mathcal{A} \to \mathcal{P}_{bf}(X)$, we have $\mu(A) = \mu(\bigcup_{s=1}^{\infty} \bigcap_{k=s}^{\infty} A_{n_k}) = \mu(\liminf_k A_{n_k})$, that is, μ has property (PS).

iv) $\Rightarrow$ ii) Consider arbitrary $A \in \mathcal{A}$, $(B_n) \subset \mathcal{A}, C \in A \cap \mathcal{A}$, so that $\lim_{n\to\infty} h(\mu(A \cap B_n), \mu(A)) = 0$. There exists a subsequence $(B_{n_k})_k$ of (B_n) so that $\limsup_n h(\mu(C \cap B_n), \mu(C)) = \lim_{k\to\infty} h(\mu(C \cap B_{n_k}), \mu(C))$.

Applying property (PS) for $A \cap B_{n_k}$ and since $\mu : \mathcal{A} \to \mathcal{P}_{bf}(X)$, there exists $(B_{n_{k_s}} \cap A)_s$ so that $\mu(\liminf_s (B_{n_{k_s}} \cap A)) = \mu(A)$.

Consequently, by the pseudo-null-additivity of μ, according to Remark 2.4, we have $\mu((\liminf_s(B_{n_{k_s}} \cap A) \cap C))) = \mu(A \cap C)$, that is, equivalently, $\mu(\liminf_s(B_{n_{k_s}} \cap C)) = \mu(C)$. On the other hand, because $\bigcap_{t=s}^{\infty} C \cap B_{n_{k_t}} \underset{s\to\infty}{\nearrow} \liminf_s(B_{n_{k_s}} \cap C)$ and μ is continuous from below, we have

$$0 \le \limsup_n h(\mu(C \cap B_n), \mu(C)) = \lim_{s\to\infty} h(\mu(C \cap B_{n_{k_s}}), \mu(C))$$

$$= \lim_{s\to\infty} e(\mu(C), \mu(C \cap B_{n_{k_s}})) \le \lim_{s\to\infty} [e(\mu(C), \mu(\bigcap_{t=s}^{\infty}(C \cap B_{n_{k_t}})))$$

$$+ e(\mu(\bigcap_{t=s}^{\infty}(C \cap B_{n_{k_t}})), \mu(C \cap B_{n_{k_s}}))] = \lim_{s\to\infty} e(\mu(C), \mu(\bigcap_{t=s}^{\infty}(C \cap B_{n_{k_t}})))$$

$$\le \lim_{s\to\infty} [e(\mu(C), \mu(\bigcup_{s=1}^{\infty}\bigcap_{t=s}^{\infty}(C \cap B_{n_{k_t}})))$$

$$+ e(\mu(\bigcup_{s=1}^{\infty}\bigcap_{t=s}^{\infty}(C \cap B_{n_{k_t}})), \mu(\bigcap_{t=s}^{\infty} C \cap B_{n_{k_t}}))]$$

$$= e(\mu(C), \mu(\liminf_s(B_{n_{k_s}} \cap C))) = 0,$$

so $\limsup_n h(\mu(C \cap B_n), \mu(C)) = 0$, whence $\lim_{n\to\infty} h(\mu(C \cap B_n), \mu(C)) = 0$, that is, μ is pseudo-autocontinuous from below.

iv) $\Rightarrow$ v) Consider arbitrary $A \in \mathcal{A}$, $(A_n)_n \subset A \cap \mathcal{A}$, with $\lim_{n\to\infty} h(\mu(A_n), \mu(A)) = 0$. Since $\mu : \mathcal{A} \to \mathcal{P}_{bf}(X)$ has (PS), there exists a subsequence $(A_{n_k})_k \subset (A_n)$ so that $\mu(A) = \mu(\underline{\lim}_k A_{n_k}) = \mu(A \backslash (A \backslash \underline{\lim}_k A_{n_k}))$.

Since μ is pseudo-null-additive, then $\mu((A \backslash \underline{\lim}_k A_{n_k}) \cup C)) = \mu(C)$, for every $C \in A \cap \mathcal{A}$. Particularly, for $C = \emptyset$, we get $\mu(A \backslash \underline{\lim}_k A_{n_k}) = \{0\}$.

v) $\Rightarrow$ iv) Consider arbitrary $A \in \mathcal{A}$, $(A_n)_n \subset A \cap \mathcal{A}$, with $\lim_{n\to\infty} h(\mu(A_n), \mu(A)) = 0$.

By the hypothesis, there exists a subsequence $(A_{n_k})_k \subset (A_n)$ so that $\mu(A \backslash \underline{\lim}_k A_{n_k}) = \{0\}$. Since μ is pseudo-null-additive, then $\mu((A \backslash \underline{\lim}_k A_{n_k}) \cup C)) = \mu(C)$, for every $C \in A \cap \mathcal{A}$. Particularly, for $C = \underline{\lim}_k A_{n_k}$, we get $\mu(A) = \mu(\underline{\lim}_k A_{n_k})$.

It only remains to prove that v) $\Rightarrow$ iii). For this, one can use the same method as in the implication iv) $\Rightarrow$ ii), and Remark 2.4. $\square$

Now, we are in position to give the following pseudo-version of Riesz's theorem:

Theorem 4.5 (a pseudo-version of Riesz's theorem)**.** *Let be μ a fuzzy multimeasure, $A \in \mathcal{A}$, $f \in \mathcal{M}$ and $\{f_n\} \subset \mathcal{M}$ so that $f_n \xrightarrow[A]{p.\mu} f$. Then there exists a subsequence $\{f_{n_k}\}_k$ of $\{f_n\}$ such that $f_{n_k} \xrightarrow[A]{p.a.e.} f$ if and only if μ has property (PS).*

Proof. *Necesssity.* Let be $(A_n)_n \subset \mathcal{A}$ such that $\lim_{n\to\infty} h(\mu(A\backslash A_n), \mu(A)) = 0$ and let us define for every $n \in \mathbb{N}$,

$$f_n(x) = \begin{cases} 1, & \text{if } x \in A_n \\ 0, & \text{if } x \in A\backslash A_n. \end{cases}.$$

We see that $\{f_n\} \subset \mathcal{M}$ and $f_n \xrightarrow[A]{p.\mu} 0$. Then there exists a subsequence $\{f_{n_k}\}_k$ of $\{f_n\}$ such that $f_{n_k} \xrightarrow[A]{p.a.e.} 0$, whence $\mu(A) = \mu(A\backslash \bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty}\{x \in A; |f_{n_k}(x) - 0| \geq \varepsilon\})$, for every $\varepsilon > 0$.

Particularly, for $\varepsilon = \frac{1}{2}$, we obtain $\mu(A) = \mu(A\backslash \bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} A_{n_k})$, or, equivalently, $\mu(A) = \mu(\bigcup_{m=1}^{\infty} \bigcap_{k=m}^{\infty}(A\backslash A_{n_k}))$, which says that μ has property (PS).

Sufficiency. Suppose that μ has property (PS) and $f_n \xrightarrow[A]{p.\mu} f$. Then we can find $\{n_k\}_k$ such that $\lim_{n\to\infty} h(\mu(A), \mu(A\backslash B_k)) = 0$, where $B_k = \{x \in A; |f_{n_k}(x) - f(x)| \geq \frac{1}{k}\}$, for $k \in \mathbb{N}^*$.

Since μ has property (PS), there exists a subsequence $(B_{k_l})_k$ of $(B_k)_k$ such that $\mu(A) = \mu(A\backslash \bigcap_{m=1}^{\infty} \bigcup_{l=m}^{\infty} B_{k_l})$. It is easy to see that $\{f_{n_{k_m}}\}_m$ converges to f on the set $A\backslash \bigcap_{m=1}^{\infty} \bigcup_{l=m}^{\infty} B_{k_l}$, and, consequently, $f_{n_{k_m}} \xrightarrow[A]{p.a.e.} f$. $\square$

From Theorem 4.5 and by Theorem 4.3 from [22], we obtain:

Corollary 4.6. *If μ is a fuzzy multimeasure which satisfies properties (PS) and (PE), $A \in \mathcal{A}$, $f \in \mathcal{M}$ and $\{f_n\} \subset \mathcal{M}$ are so that $f_n \xrightarrow[A]{p.\mu} f$, then there exists a subsequence $\{f_{n_k}\}_k$ of $\{f_n\}$ such that $f_{n_k} \xrightarrow[A]{p.a.u.} f$.*

5. Concluding remarks

In this paper, we investigated for set-valued non-additive monotonic set functions, some relationships among the main types of convergences of sequences of measurable functions.

In this way, we insisted on different types of pseudo-convergences of sequences of measurable functions, such as, pseudo-almost everywhere (p.a.e.), pseudo-almost uniform (p.a.u) convergences and pseudo-convergence in measure (p.μ) and on the relationships among them, or with almost everywhere, almost uniform convergences and convergence in measure.

Thus, we obtained several set-valued versions of Lebesgue theorem and a pseudo-version of Riesz's theorem.

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