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An extension of Petrović's inequality for $h-$convex ($h-$concave) functions in plane

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Abstract: In this paper, Petrović's inequality is generalized for $h-$convex functions on coordinates with the condition that h is supermultiplicative. In the case, when h is submultiplicative, Petrović's inequality is generalized for $h-$concave functions. Also particular cases for $P-$function, Godunova-Levin functions, $s-$Godunova-Levin functions and $s-$convex functions has been discussed.

Keywords: Petrović's inequality, $h-$convex functions, $h-$concave functions, $h-$convex functions on coordinates, $h-$concave functions on coordinates.

MSC: Primary 26A51; Secondary 26D15.

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1. Introduction

Let $h : [c, d] \to \mathbb{R}$ be a non-negative function and $(0, 1) \subseteq [c, d]$. A function $f : [a, b] \to \mathbb{R}$ is said to be an $h-$convex, if f is non-negative for all $x, y \in [a, b]$ and $\alpha \in (0, 1)$, one has

$$f(\alpha x + (1-\alpha)y) \geq h(\alpha)f(x) + h(1-\alpha)f(y). \tag{1}$$

If above inequality is reversed, then f is said to be h-concave.

The $h-$convex function was introduced by Varošanec in [1]. This function generalized convex function and many other generalization of convex function like $s-$convex function, Godunova-Levin function, $s-$Godunova-Levin function and $P-$function given in [1–3].

Remark 1. Particular value of h in inequality (1) gives us the following results:

1. $h(\alpha) = \alpha$ gives the convex functions.
2. $h(\alpha) = 1$ gives the $P-$functions.
3. $h(\alpha) = \alpha^s$ and $\alpha \in (0, 1)$ gives the $s-$convex functions of second sense.
4. $h(\alpha) = \frac{1}{\alpha}$ and $\alpha \in (0, 1)$ gives the Godunova-Levin functions.
5. $h(\alpha) = \frac{1}{\alpha^s}$ and $\alpha \in (0, 1)$ gives the $s-$Godunova-Levin functions of second sense.

In case of $h-$concavity, following results are valid:

6. $h(\alpha) = 1$ gives the reverse $P-$functions.
7. $h(\alpha) = \frac{1}{\alpha}$ gives the reverse Godunova-Levin functions.
8. $h(\alpha) = \frac{1}{\alpha^s}$ gives the reverse $s-$Godunova-Levin functions of second sense.

In [4], Dragomir gave the definition of convex functions on coordinates. Following his idea, the $h-$convex on coordinates was introduced by Alomari *et al.* in [5].

Definition 1. Let $\Delta = [a_1, b_1] \times [a_2, b_2] \subseteq \mathbb{R}^2$ and $f : \Delta \to \mathbb{R}$ be a mapping. Define partial mappings

$$f_y : [a_1, b_1] \to \mathbb{R} \text{ by } f_y(u) = f(u, y) \tag{2}$$

and
$$f_x : [a_2, b_2] \to \mathbb{R} \text{ by } f_x(v) = f(x, v). \tag{3}$$

Also let interval $[c, d]$ contains $(0, 1)$ and $h : [c, d] \to \mathbb{R}$ be a positive function. A mapping $f : \Delta \to \mathbb{R}$ is said to be $h-$convex ($h-$concave) on Δ, if the partial mappings defined in (2) and (3) are $h-$convex ($h-$concave) on $[a, b]$ and $[c, d]$ respectively for all $y \in [c, d]$ and $x \in [a, b]$.

Remark 2. From above definition, one can deduce the definitions of those particular cases on coordinates.

In [6] (also see [7, p. 154]), Petrović proved the following result, which is known as Petrović's inequality in the literature.

Theorem 2. *Suppose that $(x_1, ..., x_n)$ and $(p_1, ..., p_n)$ be non-negative n-tuples such that $\sum_{k=1}^{n} p_k x_k \geq x_i$ for $i = 1, ..., n$ and $\sum_{k=1}^{n} p_k x_k \in [0, a]$. If f is a convex function on $[0, a]$, then the inequality*

$$\sum_{k=1}^{n} p_k f(x_k) \leq f\left(\sum_{k=1}^{n} p_k x_k\right) + \left(\sum_{k=1}^{n} p_k - 1\right) f(0) \tag{4}$$

is valid.

A function $h : [c, d] \to \mathbb{R}$ is said to be a submultiplicative function if

$$h(xy) \leq h(x)h(y), \tag{5}$$

for all $x, y \in [c, d]$. If the above inequality is reversed, then h is said to be supermultiplicative function. If equality holds in the above inequality, then h is said to be multiplicative function.

By considering h to be supermultiplicative along with other condition, in the following theorem generalization of Petrović's inequality was proved by Rehman *et al.* in [8].

Theorem 3. *Let $(x_1, ..., x_n)$ be non-negative n-tuples and $(p_1, ..., p_n)$ be positive n-tuples such that*

$$\sum_{k=1}^{n} p_k x_k \in [0, a] \text{ and } \sum_{k=1}^{n} p_k x_k \geq x_j \text{ for each } j = 1, ..., n. \tag{6}$$

Also let $h : [0, \infty) \to \mathbb{R}^+$ be a supermultiplicative function such that

$$h(\alpha) + h(1 - \alpha) \leq 1, \text{ for all } \alpha \in (0, 1). \tag{7}$$

If $f : [0, \infty) \to \mathbb{R}$ be an $h-$convex function on $[0, \infty)$, then

$$\sum_{j=1}^{n} p_j f(x_j) \leq \frac{\sum_{j=1}^{n} p_j h(x_j - c)}{h\left(\sum_{k=1}^{n} p_k x_k - c\right)} f\left(\sum_{k=1}^{n} p_k x_k\right) + \left(\sum_{j=1}^{n} p_j - \frac{\sum_{j=1}^{n} p_j h(x_j - c)}{h\left(\sum_{k=1}^{n} p_k x_k - c\right)}\right) f(c). \tag{8}$$

The following reverse version of above theorem was also proved in [8].

Theorem 4. *Let $(x_1, ..., x_n)$ be non-negative n-tuples and $(p_1, ..., p_n)$ be positive n-tuples and the conditions given in (6) are valid. Also let $h : [0, a] \to \mathbb{R}^+$ be a submultiplicative function such that*

$$h(\alpha) + h(1 - \alpha) \geq 1, \text{ for all } \alpha \in (0, 1). \tag{9}$$

If $f : [0, a] \to \mathbb{R}$ be an $h-$concave function on $[0, a]$, then reverse of (8) is valid.

In recent years, $h-$Convex functions are considered in literature by many researchers and mathematicians, for example, see [1,3,5,9] and references there in. Many authors worked on Petrović's inequality by giving results related to it, for example see [6,7,10] and it has been generalized for $m-$convex functions by Bakula *et al.* in [11]. In [12], Petrović's inequality was generalized on coordinates by using the definition of convex functions on coordinates.

In this paper, Petrović's inequality is generalized for $h-$convex functions on coordinates, when h is supermultiplicative function. When h is submultiplicative, Petrović's inequality is generalized for $h-$concave functions on coordinates.

2. Main results

The following theorem consist the result for generalized Petrović's inequality for $h-$convex functions on coordinates.

Theorem 5. *Let $(x_1, ..., x_n)$ and $(y_1, ..., y_n)$ be non-negative n-tuples, $(p_1, ..., p_n)$ and $(q_1, ..., q_n)$ be positive n-tuples such that*

$$\sum_{k=1}^{n} p_k x_k \in [0, a], \sum_{k=1}^{n} p_k x_k \geq x_j \text{ for each } j = 1, ..., n, \tag{10}$$

and

$$\sum_{j=1}^{n} q_j y_j \in [0, b], \sum_{j=1}^{n} q_j y_j \geq y_i \text{ for each } i = 1, ..., n. \tag{11}$$

Also let $h : [0, \infty) \to \mathbb{R}^+$ be a supermultiplicative function such that (7) is valid. If $f : [0, a] \times [0, b] \to \mathbb{R}$ be an $h-$convex function on coordinates, then

$$\begin{aligned}\sum_{k=1}^{n}\sum_{j=1}^{n} p_k q_j f(x_k, y_j) \leq\ & \frac{\sum_{j=1}^{n} p_j h(x_j - c_1)}{h\left(\sum_{k=1}^{n} p_k x_k - c_1\right)} \left\{ \frac{\sum_{j=1}^{n} q_j h(y_j - c_2)}{h\left(\sum_{k=1}^{n} q_k y_k - c_2\right)} f\left(\sum_{k=1}^{n} p_k x_k, \sum_{j=1}^{n} q_j y_j\right)\right. \\ & \left. + \left(\sum_{j=1}^{n} q_j - \frac{\sum_{j=1}^{n} q_j h(y_j - c_2)}{h\left(\sum_{k=1}^{n} q_k y_k - c_2\right)}\right) f\left(\sum_{k=1}^{n} p_k x_k, c_2\right)\right\} + \left(\sum_{j=1}^{n} p_j - \frac{\sum_{j=1}^{n} p_j h(x_j - c_1)}{h\left(\sum_{k=1}^{n} p_k x_k - c_1\right)}\right) \\ & \left\{ \frac{\sum_{j=1}^{n} q_j h(y_j - c_2)}{h\left(\sum_{k=1}^{n} q_k y_k - c_2\right)} f\left(c_1, \sum_{j=1}^{n} q_j y_j\right) + \left(\sum_{j=1}^{n} q_j - \frac{\sum_{j=1}^{n} q_j h(y_j - c_2)}{h\left(\sum_{k=1}^{n} q_k y_k - c_2\right)}\right) f(c_1, c_2)\right\},\end{aligned} \tag{12}$$

where $x_i > c_1, y_j > c_2$.

Proof. Let $f_x : [0, a] \to \mathbb{R}$ and $f_y : [0, b] \to \mathbb{R}$ be mappings such that $f_x(v) = f(x, v)$ and $f_y(u) = f(u, y)$. Since f is coordinated $h-$convex on $[0, a] \times [0, b]$, therefore f_y is $h-$convex on $[0, b]$, so by Theorem 3, one has

$$\sum_{j=1}^{n} p_j f_y(x_j) \leq \frac{\sum_{j=1}^{n} p_j h(x_j - c_1)}{h\left(\sum_{k=1}^{n} p_k x_k - c_1\right)} f_y\left(\sum_{k=1}^{n} p_k x_k\right) + \left(\sum_{j=1}^{n} p_j - \frac{\sum_{j=1}^{n} p_j h(x_j - c_1)}{h\left(\sum_{k=1}^{n} p_k x_k - c_1\right)}\right) f_y(c_1).$$

This is equivalent to

$$\sum_{j=1}^{n} p_j f(x_j, y) \leq \frac{\sum_{j=1}^{n} p_j h(x_j - c_1)}{h\left(\sum_{k=1}^{n} p_k x_k - c_1\right)} f\left(\sum_{k=1}^{n} p_k x_k, y\right) + \left(\sum_{j=1}^{n} p_j - \frac{\sum_{j=1}^{n} p_j h(x_j - c_1)}{h\left(\sum_{k=1}^{n} p_k x_k - c_1\right)}\right) f(c_1, y),$$

by setting $y = y_j$, we get

$$\sum_{j=1}^{n} p_j f(x_j, y_j) \leq \frac{\sum\limits_{j=1}^{n} p_j h(x_j - c_1)}{h\left(\sum\limits_{k=1}^{n} p_k x_k - c_1\right)} f\left(\sum_{k=1}^{n} p_k x_k, y_j\right) + \left(\sum_{j=1}^{n} p_j - \frac{\sum\limits_{j=1}^{n} p_j h(x_j - c_1)}{h\left(\sum\limits_{k=1}^{n} p_k x_k - c_1\right)}\right) f(c_1, y_j).$$

Multiplying above inequality by p_j and taking sum for $j = 1, ..., n$, one has

$$\sum_{k=1}^{n}\sum_{j=1}^{n} p_j q_j f(x_j, y_j) \leq \frac{\sum\limits_{j=1}^{n} p_j h(x_j - c_1)}{h\left(\sum\limits_{k=1}^{n} p_k x_k - c_1\right)} \sum_{k=1}^{n} q_j f\left(\sum_{k=1}^{n} p_k x_k, y_j\right) + \left(\sum_{j=1}^{n} p_j - \frac{\sum\limits_{j=1}^{n} p_j h(x_j - c_1)}{h\left(\sum\limits_{k=1}^{n} p_k x_k - c_1\right)}\right) \sum_{k=1}^{n} q_j f(c_1, y_j). \tag{13}$$

Now again by Theorem 4, one has

$$\sum_{j=1}^{n} q_j f\left(\sum_{k=1}^{n} p_k x_k, y_j\right) \leq \frac{\sum\limits_{j=1}^{n} q_j h(y_j - c_2)}{h\left(\sum\limits_{k=1}^{n} q_k y_k - c_2\right)} f\left(\sum_{k=1}^{n} p_k x_k, \sum_{j=1}^{n} q_j y_j\right) + \left(\sum_{j=1}^{n} q_j - \frac{\sum\limits_{j=1}^{n} q_j h(y_j - c_2)}{h\left(\sum\limits_{k=1}^{n} q_k y_k - c_2\right)}\right) f\left(\sum_{k=1}^{n} p_k x_k, c_2\right)$$

and

$$\sum_{j=1}^{n} q_j f(c_1, y_j) \leq \frac{\sum\limits_{j=1}^{n} q_j h(y_j - c_2)}{h\left(\sum\limits_{k=1}^{n} q_k y_k - c_2\right)} f\left(c_1, \sum_{j=1}^{n} q_j y_j\right) + \left(\sum_{j=1}^{n} q_j - \frac{\sum\limits_{j=1}^{n} q_j h(y_j - c_2)}{h\left(\sum\limits_{k=1}^{n} q_k y_k - c_2\right)}\right) f(c_1, c_2).$$

Putting these values in inequality (13), we get the required result. □

In the following theorem, we give the Petrović's inequality for $h-$convex functions on coordinates.

Theorem 6. *Let the conditions given in Theorem 5 are valid. If $f : [0, a] \times [0, b] \to \mathbb{R}$ be an $h-$convex function on coordinates, then*

$$\begin{aligned}
&\sum_{k=1}^{n}\sum_{j=1}^{n} p_j q_j f(x_j, y_j) \\
&\leq \frac{\sum\limits_{j=1}^{n} p_j h(x_j)}{h\left(\sum\limits_{k=1}^{n} p_k x_k\right)} \left\{ \frac{\sum\limits_{j=1}^{n} q_j h(y_j)}{h\left(\sum\limits_{k=1}^{n} q_k y_k\right)} f\left(\sum_{k=1}^{n} p_k x_k, \sum_{j=1}^{n} q_j y_j\right) + \left(\sum_{j=1}^{n} q_j - \frac{\sum\limits_{j=1}^{n} q_j h(y_j)}{h\left(\sum\limits_{k=1}^{n} q_k y_k\right)}\right) f\left(\sum_{k=1}^{n} p_k x_k, 0\right) \right\} \\
&+ \left(\sum_{j=1}^{n} p_j - 1\right) \left\{ \frac{\sum\limits_{j=1}^{n} q_j h(y_j)}{h\left(\sum\limits_{k=1}^{n} q_k y_k\right)} f\left(0, \sum_{j=1}^{n} q_j y_j\right) + \left(\sum_{j=1}^{n} q_j - \frac{\sum\limits_{j=1}^{n} q_j h(y_j)}{h\left(\sum\limits_{k=1}^{n} q_k y_k\right)}\right) f(0, 0) \right\}.
\end{aligned} \tag{14}$$

Proof. If we take $c_1 = 0 = c_2$ in Theorem 5, we get the required result. □

In the following corollary, we give the Petrović's inequality for convex functions on coordinates which is given in [12].

Theorem 7. *Let the conditions given in Theorem 5 are valid. If $f : [0, a] \times [0, b] \to \mathbb{R}$ be a convex function on coordinates, then*

$$\begin{aligned}
\sum_{k=1}^{n}\sum_{j=1}^{n} p_j q_j f(x_j, y_j) \quad \leq \quad & f\left(\sum_{k=1}^{n} p_k x_k, \sum_{j=1}^{n} q_j y_j\right) + \left(\sum_{j=1}^{n} q_j - 1\right) f\left(\sum_{k=1}^{n} p_k x_k, 0\right) \\
&+ \left(\sum_{j=1}^{n} p_j - 1\right) \left\{ f\left(0, \sum_{j=1}^{n} q_j y_j\right) + \left(\sum_{j=1}^{n} q_j - 1\right) f(0, 0) \right\}.
\end{aligned} \tag{15}$$

Proof. If we take $h(x) = x$ for all $x \in [0, \infty)$, then it satisfied the condition imposed on h given in Theorem 6. Hence using this value of h in above theorem gives the required result. □

One can see that the condition on function h given in (7) restrict us to give Petrović's type inequalities for particular cases of $h-$convex functions given in Remark 1. If we consider reverse inequality in (7), then it covers some of particular cases but for $h-$concave function.

In the following theorem, reverse of (12) has been concluded. The notable thing is the requirements of submultiplicity and reverse of (7) for function h along with h-concavity of the function f.

Theorem 8. *Let $(x_1, ..., x_n)$ and $(y_1, ..., y_n)$ be non-negative n-tuples, $(p_1, ..., p_n)$ and $(q_1, ..., q_n)$ be positive $n-$tuples such that (10) and (11) are valid. Also let $h : [0, \infty) \to \mathbb{R}^+$ be a submultiplicative function such that (9) is valid. If $f : [0, a] \times [0, b] \to \mathbb{R}$ be an $h-$concave function on coordinates, then the reverse of inequality (12) holds.*

Proof. By using Theorem (4) and following the steps of Theorem 5, one can deduce the required results. □

In the following theorem, we give the Petrović's inequality for $h-$concave functions on coordinates.

Theorem 9. *Let the conditions given in Theorem 8 are valid.*

Also let $h : [0, \infty) \to \mathbb{R}^+$ be a submultiplicative function. If $f : [0, a] \times [0, b] \to \mathbb{R}$ be an $h-$concave function on coordinates, then the reverse of inequality (14) is valid.

Proof. If we take $c_1 = 0 = c_2$ in Theorem 8, we get the required result. □

In the following theorem, we give the Petrović's inequality for concave functions on coordinates.

Theorem 10. *Let the conditions given in Theorem 8 are valid. If $f : [0, a] \times [0, b] \to \mathbb{R}$ be a concave function on coordinates, then then the reverse of inequality (15) is valid.*

Proof. If we take $h(x) = x$ and $c_1 = 0 = c_2$ in Theorem 8, we get the required result.

Theorem 11. *Let $(x_1, ..., x_n)$ and $(y_1, ..., y_n)$ be non-negative n-tuples, $(p_1, ..., p_n)$ and $(q_1, ..., q_n)$ be positive n-tuples such that (10) and (11) are valid. If $f : [0, a] \times [0, b] \to \mathbb{R}$ is reverse $P-$function on coordinates, then*

$$\sum_{k=1}^{n} \sum_{j=1}^{n} p_k q_j f(x_k, y_j) \leq \sum_{i=1}^{n} \sum_{j=1}^{n} p_i q_j \left(\sum_{k=1}^{n} p_k x_k, \sum_{j=1}^{n} q_j y_j \right). \tag{16}$$

□

Remark 3. Consider $h(x) = \frac{1}{x}$, then $h(\alpha) + h(1-\alpha) = \frac{1}{\alpha} + \frac{1}{1-\alpha} > 1$ for all $\alpha \in (0, 1)$. Using above value of h in Theorem 8 gives Petrović type inequality for reverse Godunova-Levin functions on coordinates.

Remark 4. Let us consider $H(h) = h(\alpha) + h(1-\alpha) - 1, \alpha \in (0, 1)$, we take $g_1(\alpha) := H(\alpha^s) = \alpha^s + (1-\alpha)^s - 1$, where $s \in (0, 1)$. In [8], it has been shown that g_1 is positive by considering different values of α and s in interval $(0, 1)$, therefore $h(\alpha) = \alpha^s$ for $\alpha, s \in (0, 1)$ satisfied the conditions of Theorem 8, but it doesn't satisfies the conditions of Theorem 5. Hence the above value of h in Theorem 8 leads us to the Petrović type inequalities for reverse of $s-$Godunova-Levin on coordinates.

Remark 5. Let us consider $g_2(\alpha) := H\left(\frac{1}{\alpha^s}\right) = \frac{1}{\alpha^s} + \frac{1}{(1-\alpha)^s} - 1$, where $s \in (0, 1)$. This function is also discussed in [8] and it has been shown that g_2 is positive for different values of α and s in $(0, 1)$. Thus it satisfied the conditions of Theorem 8, but it doesn't satisfy the conditions of Theorem 5. Hence the above value of h in Theorem 8 leads us to the Petrović type inequalities for $s-$concave function on coordinates.

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