Fractional Fourier Series with Separation of Variables Technique and Its Application on Fractional differential Equations.

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Abstract: When using some classical methods, such us separation of variables; it is impossible to find a general solution for some differential equations. Therefore, we suggest adding conformable fractional Fourier series to get a new technique to solve fractional Benjamin Bana Mahony and Heat Equations. Furtheremore, we give new numerical approximation for functions using mathematica coding called conformable fractional Fourier series approximation.

Key–Words: Fractional fourier series, Conformable fractional derivative.

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1 Introduction

Fractional Fourier transform is one of the most important tools in applied sciences [1]. It has been proved that we can solve partial differential equations using Fractional Fourier transform [2]. Fractional partial differential equations also appeared to have many applications in physics and engineering. See [1], [3], [4], [5], [6], [7] and [8]. There are many definitions of fractional derivative. One of the most recent ones is the conformable fractional derivative [9].

For more applications on conformable fractional derivative we refer the reader to [10], [11], [12], [13], [14] and [15].

Recently [16], fractional Taylor power series was introduced, and a beautiful theory was layed there. However, no work is done on fractional Fourier series, though there is some work on fractional fourier transform.

The aim of this paper is to introduce conformable fractional Fourier series with separation of variables as a new technique to help us solve some fractional differential equations (Benjamin Bana Mahony and Heat Equations). These equations can not be solved using classical methods. Moreover, we create a new numerical approximation for functions using the mathematica coding called conformable fractional Fourier series approximation.

2 Basics of Conformable Fractional Derivative

A definition called conformable fractional derivative was introduced.

Definition 1 *Let $\alpha > 0$, then we define the fractional derivative of f of order α as*

$$T_\alpha(f)(t) = \lim_{\epsilon \to 0} \frac{f(t+\epsilon t^{1-\alpha}) - f(t)}{\epsilon}.$$

for all $t > 0$ and $\alpha \in (0,1)$. We shall write $f^\alpha(t)$ for $T_\alpha(f)(t)$.

One can easily show that T_α satisfies all the properties in the following theorem.

Theorem 2 *Let $\alpha \in (0,1]$ and f, g be $\alpha-$differentiable at a point $t > 0$. Then*
(1) $T_\alpha(af+bg) = aT_\alpha(f)+bT_\alpha(g)$, *for all $a, b \in R$.*
(2) $T_\alpha(t^p) = pt^{p-\alpha}$, *for all $p \in R$.*
(3) $T_\alpha(\lambda) = 0$, *for all constant functions $f(t) = \lambda$.*
(4) $T_\alpha(fg) = gT_\alpha(f) + fT_\alpha(g)$.
(5) $T_\alpha(\frac{f}{g}) = \frac{gT_\alpha(f)-fT_\alpha(g)}{g^2}$.
(6) *In addition, if f is differentiable, then* $T_\alpha(f)(t) = t^{1-\alpha}\frac{df}{dt}(t)$.

The conformable fractional derivatives of certain functions:
(1) $T_\alpha(\sin \frac{1}{\alpha}t^\alpha) = \cos \frac{1}{\alpha}t^\alpha$.
(2) $T_\alpha(\cos \frac{1}{\alpha}t^\alpha) = -\sin \frac{1}{\alpha}t^\alpha$.

(3) $T_\alpha(e^{\frac{1}{\alpha}t^\alpha}) = e^{\frac{1}{\alpha}t^\alpha}$.
(4) $T_\alpha(t^p) = pt^{p-\alpha}$, for all $p \in R$.

On letting $\alpha = 1$ in these derivatives, we get the corresponding ordinary derivatives.

One should notice that a function could be $\alpha-$differentiable at a point but not differentiable, for example, take $f(t) = 2\sqrt{t}$.
Then $T_{\frac{1}{2}}(f)(0) = \lim_{t\to 0^+} T_{\frac{1}{2}}(f)(t) = 1$.
Where $T_{\frac{1}{2}}(f)(t) = 1$, for $t > 0$.
This is not the case for the known classical fractional derivatives since $T_1(f)(0)$ does not exist.

3 Fractional Fourier Series

Let $0 < \alpha \leq 1$, and $\varphi : [0,\infty) \to R$ be defined by $\varphi(t) = \frac{t^\alpha}{\alpha}$ and $g : [0,\infty) \to R$ be any function.
Let $f : [0,\infty) \to R$ be defined by $f(t) = g(\varphi(t))$.
For example, if $g(t) = \cos\ t$, then $f(t) = \cos(\frac{t^\alpha}{\alpha})$.

Definition 3 *A function $f(t)$ is called $\alpha-$periodical with period p if*

$$f(t) = g(\varphi(t)) = g(\varphi(t) + \frac{p^\alpha}{\alpha}),\ \ for\ all\ t \in [0,\infty).$$

As an example, $f(t) = \cos(\frac{t^\alpha}{\alpha})$ is $\alpha-$periodic with period $p = (2\alpha\pi)^{\frac{1}{\alpha}}$.

Definition 4 *Two functions f, h are called $\alpha-$orthogonal on $[0,b]$ if*

$$\int_0^b \frac{f(t)h(t)}{t^{1-\alpha}}dt = 0.$$

Example 5 $\cos(\frac{t^\alpha}{\alpha})$ *and* $\cos(2\frac{t^\alpha}{\alpha})$ *are $\alpha-$orthogonal on* $[0,(\alpha 2\pi)^{\frac{1}{\alpha}}]$.

Proof 6 *Put the variable change $\frac{t^\alpha}{\alpha} = x$ and use property (6) in theorem 2.*
We get $dx = t^{\alpha-1}dt = \frac{dt}{t^{1-\alpha}}$.
Further, when $t = 0$, $x = 0$, and when $t = (a2\pi)^{\frac{1}{\alpha}}$, $x = 2\pi$.
Hence

$$I = \int_0^{(a2\pi)^{\frac{1}{a}}} \cos(\frac{t^a}{a})\cos(2\frac{t^a}{a})\frac{1}{t^{1-a}}dt$$

$$I = \int_0^{2\pi} \cos(x)\cos(2x)dx.$$

Using the fact that $\cos(2x) = 1 - 2\sin^2(x)$ and the variable change $u = \sin(x)$, then $du = \cos(x)dx$, when $x = 0$, $u = 0$ and when $x = 2\pi$, $u = 0$.
Then the integral I became :

$$I = \int_0^0 (1-2u^2)du = 0.$$

Hence a result as required.

In general, using the idea in Example 5; one can easily prove:

Theorem 7
(1) $\cos(n\frac{t^\alpha}{\alpha})$ *and* $\cos(m\frac{t^\alpha}{\alpha})$ *are orthogonal on* $[0,(\alpha 2\pi)^{\frac{1}{\alpha}}]$ *, for all $n \neq m$.*
(2) $\sin(n\frac{t^\alpha}{\alpha})$ *and* $\sin(m\frac{t^\alpha}{\alpha})$ *are orthogonal on* $[0,(\alpha 2\pi)^{\frac{1}{\alpha}}]$ *, for all $n \neq m$.*
(3) $\sin(n\frac{t^\alpha}{\alpha})$ *and* $\cos(m\frac{t^\alpha}{\alpha})$ *are orthogonal on* $[0,(\alpha 2\pi)^{\frac{1}{\alpha}}]$ *, for all n, m.*

Now, let us define the Fourier coefficients of an $\alpha-$periodic function with period p.

Definition 8 *Let $f : [0,\infty) \to R$ be a given peicewise continuous $\alpha-$periodic with period p: Then we define:*
(1) *The cosine $\alpha-$Fourier coefficients of f as*

$$a_n = \frac{2\alpha}{p^\alpha}\int_0^p f(t)\cos(n\frac{t^\alpha}{\alpha})\frac{dt}{t^{1-\alpha}},\ n = 0,1,2,...$$

(2) *The sine $\alpha-$Fourier coefficients of f as*

$$b_n = \frac{2\alpha}{p^\alpha}\int_0^p f(t)\sin(n\frac{t^\alpha}{\alpha})\frac{dt}{t^{1-\alpha}},\ n = 1,2,3,...$$

Example 9 *Let $f(t) = cos2\sqrt{t}$.*
The cosine and sine $\frac{1}{2}$ $-$Fourier coefficients of the function $f(t)$ is:

$$a_1 = 1,\ and\ a_n = 0\ for\ all\ n \neq 1$$

and

$$b_n = 0,\ \forall n \in N$$

where

$$p = (\alpha 2\pi)^{\frac{1}{\alpha}}\ and\ \alpha = \frac{1}{2}.$$

Now, we give the definition of the fractional Fourier series:

Definition 10 *Let $f : [0,\infty) \to R$ be a given peicewise continuous function which is $\alpha-$periodical with period p: Then the $\alpha-$fractional Fourier series of f associated with the interval $[0,p]$ is*

$$S(f)(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\ \cos(n\frac{t^\alpha}{\alpha}) + b_n\ \sin(n\frac{t^\alpha}{\alpha})$$

where a_n and b_n are as in above.

Theorem 11 *The fractional Fourier series of a piece wise continuous $\alpha-$periodical function converges pointwise to the average limit of the function at each point of discontinuity, and to the function at each point of continuity.*

Proof 12 *One can easily prove the previous classical results. See [10].*

Example 13 *Let* $f(t) = 2\sqrt{t} \ \ if \ \ 0 \leq t \leq (\frac{\pi}{2})^2$

and $f(t) = 2\sqrt{t} - 2\pi \ \ if \ \ (\frac{\pi}{2})^2 < t \leq \pi^2$.

and $\alpha = \frac{1}{2}$ *with* $p = \pi^2$ *on the interval* $[0, \pi^2]$.

Then,

$$\begin{aligned} a_0 &= \frac{\alpha}{p^\alpha}\int_0^p f(t)\frac{dt}{t^{1-\alpha}} = \frac{1}{2\pi}\int_0^{\pi^2} f(t)\frac{dt}{\sqrt{t}} \\ &= \frac{1}{2\pi}\left(\int_0^{(\frac{\pi}{2})^2} 2\sqrt{t}\frac{dt}{\sqrt{t}} + \int_{(\frac{\pi}{2})^2}^{\pi^2}(2\sqrt{t}-2\pi)\frac{dt}{\sqrt{t}}\right) \\ &= 0. \end{aligned}$$

On the other hand we have :

$$\begin{aligned} a_n &= \frac{2\alpha}{p^\alpha}\int_0^p f(t)\cos(n\frac{t^\alpha}{\alpha})\frac{dt}{t^{1-\alpha}} \\ &= \frac{1}{\pi}\int_0^{\pi^2} f(t)\cos(n2\sqrt{t}))\frac{dt}{\sqrt{t}} \\ &= \frac{1}{\pi}\int_0^{(\frac{\pi}{2})^2} 2\sqrt{t} \ \ \cos(n2\sqrt{t})\frac{dt}{\sqrt{t}} \\ &+ \frac{1}{\pi}\int_{(\frac{\pi}{2})^2}^{\pi^2}(2\sqrt{t}-2\pi)\cos(n2\sqrt{t})\frac{dt}{\sqrt{t}}. \end{aligned}$$

Using change of variables: $\theta = 2\sqrt{t}$, *we get* $d\theta = \frac{dt}{\sqrt{t}}$. *Observe that* $\theta = 0$ *if* $t = 0$, $\theta = \pi$ *if* $t = (\frac{\pi}{2})^2$, *and* $\theta = 2\pi$ *if* $t = \pi^2$, *and using integration by parts.*

Hence, the integral becomes :

$$a_n = \frac{1}{\pi}\int_0^\pi \theta\cos(n\theta)d\theta + \frac{1}{\pi}\int_\pi^{2\pi}(\theta - 2\pi)\cos(n\theta)d\theta = 0.$$

Similarly we get

$$b_n = \frac{2\alpha}{p^\alpha}\int_0^p f(t)\sin(n\frac{t^\alpha}{\alpha})\frac{dt}{t^{1-\alpha}} = \frac{2(-1)^{n+1}}{n}.$$

So

$$\begin{aligned} S_\alpha(f)(t) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \ \cos(n\frac{t^\alpha}{\alpha}) + b_n \ \sin(n\frac{t^\alpha}{\alpha}) \\ &= \sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n} \ \sin(n\frac{t^\alpha}{\alpha}). \end{aligned}$$

The figures below represent the function alongside its $\alpha-$Fractional Fourier series approximation for $10, \ 100, \ 10000$ terms where $\alpha = \frac{1}{2}$, $\alpha = \frac{1}{4}$ and $\alpha = 1$ (classical Fourier series) respectively.

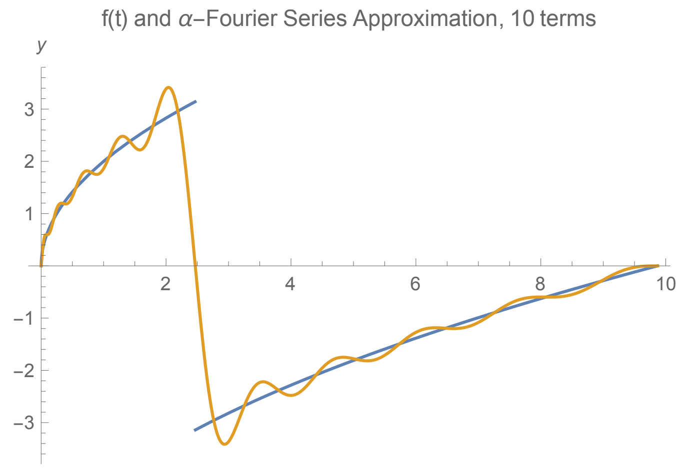

Figure 1: f(t) and alpha-Fractional Fourier Series Approximation, 10 terms.

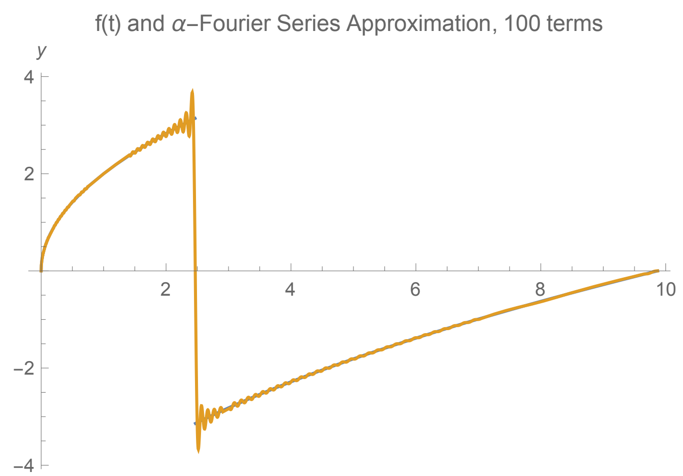

Figure 2: f(t) and alpha-Fractional Fourier Series Approximation, 100 terms.

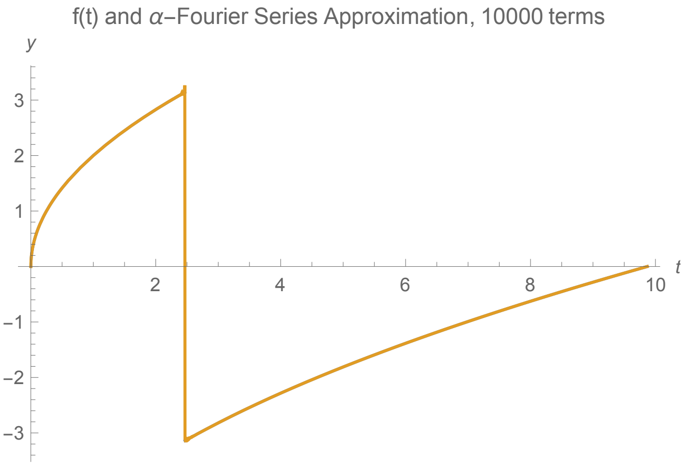

Figure 3: f(t) and alpha-Fractional Fourier Series Approximation, 10000 terms.

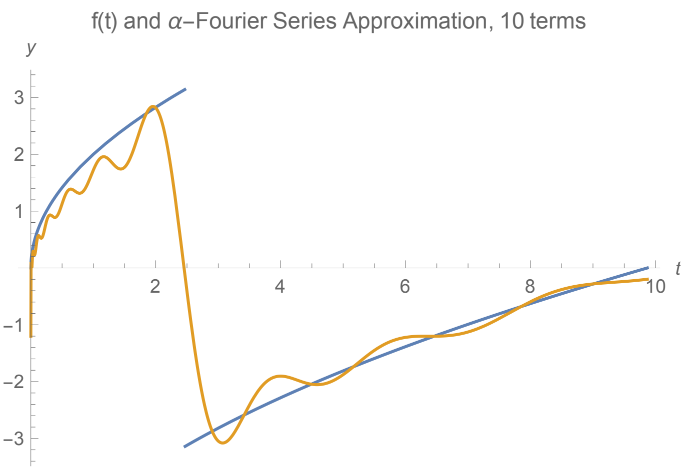

Figure 4: f(t) and alpha-Fractional Fourier Series Approximation, 10 terms.

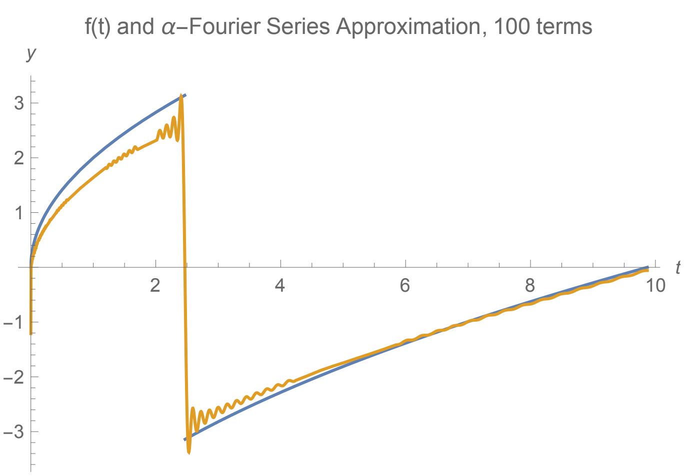

Figure 5: f(t) and alpha-Fractional Fourier Series Approximation, 100 terms.

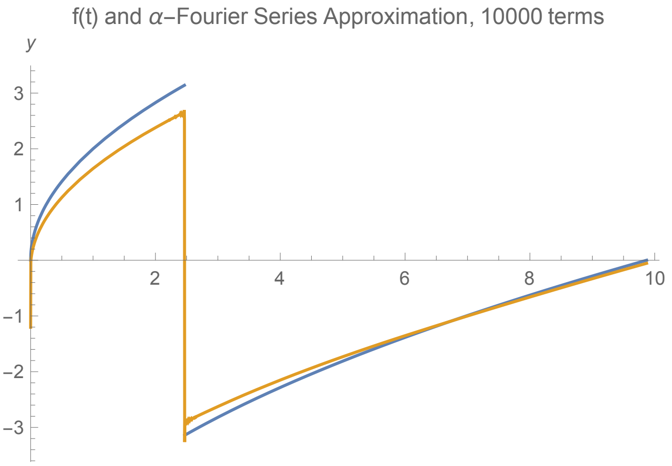

Figure 6: f(t) and alpha-Fractional Fourier Series Approximation, 10000 terms.

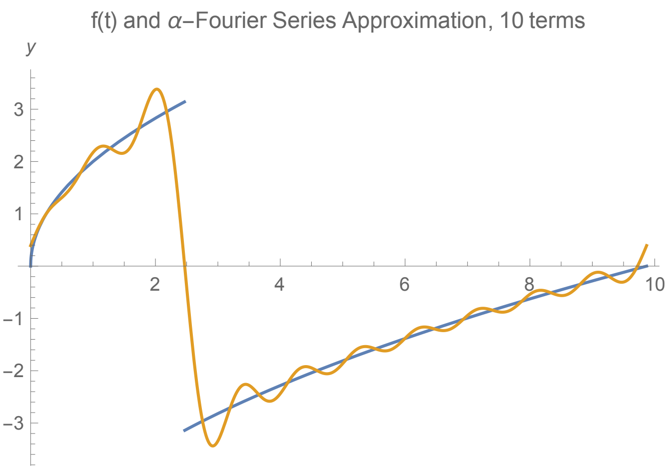

Figure 7: f(t) and Fourier Series Approximation, 10 terms.

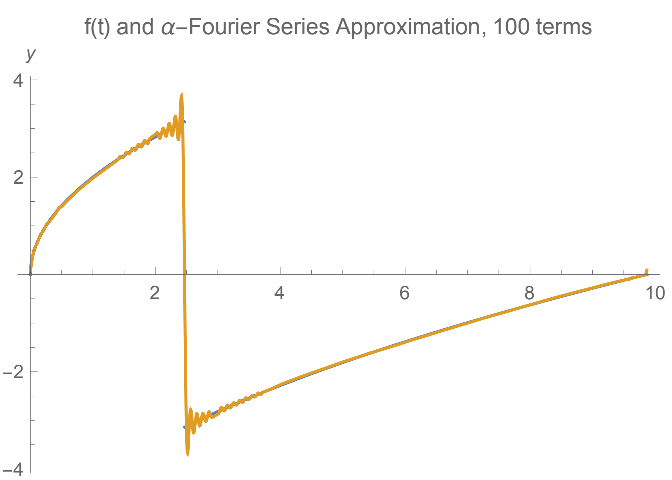

Figure 8: f(t) and Fourier Series Approximation, 100 terms.

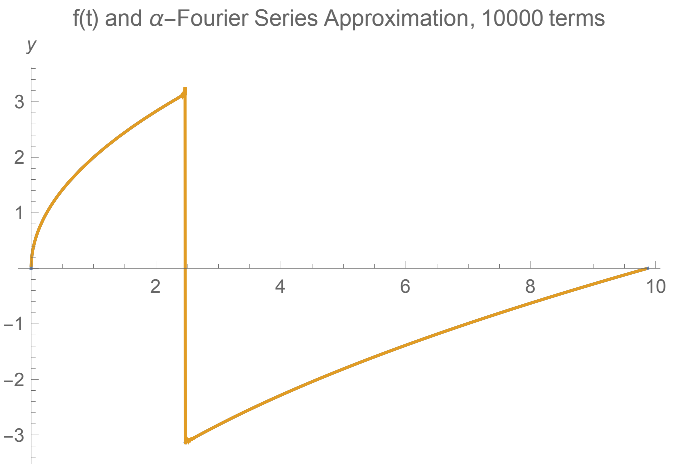

Figure 9: f(t) and Fourier Series Approximation, 10000 terms.

4 Solution of Fractional Benjamin Bana Mahony Equation

We will attempt to solve an equation called Fractional Benjamin Bana Mahony Equation using separation of variables and fractional Fourier series :

$$D_t^\beta D_t^\beta U + D_x^\alpha U = D_t^\beta D_t^\beta D_x^\alpha D_x^\alpha U. \ 0<\alpha<1 \quad (1)$$

subject to the conditions :

$$U(0,t)=0; \quad U_x^\alpha(1,t)=t^{2\beta}; \quad x>0.$$

$$U(x,0)=0; \quad U(x,1)=0; \quad 1>t>0.$$

Solution :
We will be using separation of variables technique.
Let

$$U(x,t)=P(x)Q(t).$$

Substitute in equation (1) to get :

$$P(x)Q^{2\beta}(t)+P^\alpha(x)Q(t)=P^{2\alpha}(x)Q^{2\beta}(t). \quad (2)$$

Here $P(x)$ and $Q(t)$ are the unknowns.
From which we get :

$$\frac{Q^{2\beta}(t)}{Q(t)}=\frac{P^\alpha(x)}{P^{2\alpha}(x)-P(x)}.$$

Since x and t are independent variables; then we get :

$$\frac{Q^{2\beta}(t)}{Q(t)}=\frac{P^\alpha(x)}{P^{2\alpha}(x)-P(x)}=\lambda.$$

where λ is a constant to be determined.
Hence we obtain :

$$Q^{2\beta}(t)-\lambda Q(t)=0. \quad (3)$$

$$\lambda P^{2\alpha}(x)-P^\alpha(x)-\lambda P(x)=0. \quad (4)$$

Conditions suggest that we work with equation (3) first.
Then the auxiliary equation of equation (3) is

$$r^2-\lambda=0.$$

There are three possibilities for λ :

Case (i) : $\lambda=0$:
then equation (3) becomes :

$$Q^{2\beta}(t)=0.$$

We get

$$Q(t)=c_1+c_2\frac{t^\beta}{\beta}$$

for some constants c_1 and c_2.
Conditions implies that : $c_1=c_2=0$. So

$$Q(t)=0.$$

Hence, no non trivial solution exists when $\lambda=0$.

Case (ii) : $\lambda=\mu^2>0$:
then equation (3) becomes :

$$Q^{2\beta}(t)-\mu^2Q(t)=0.$$

So the characteristic equation of equation (3) becomes $r^2-\mu^2=0$, we get two distinct solution $r_1=\mu$, $r_2=-\mu$.
Therefore the solution has the form

$$Q(t)=c_1e^{\mu\frac{t^\beta}{\beta}}+c_2e^{-\mu\frac{t^\beta}{\beta}}$$

for some constants c_1 and c_2.
But again conditions show that $Q(0)=c_1+c_2=0$, $c_1=-c_2$, also $Q(1)=c_1e^{\frac{\mu}{\beta}}+c_2e^{\frac{-\mu}{\beta}}=0$.
So, $c_1(e^{\frac{\mu}{\beta}}-e^{-\frac{\mu}{\beta}})=0$, thus $c_1=c_2=0$.
Hence, no non trivial solution exists when $\lambda=\mu^2$.

Case (iii) : $\lambda=-\mu^2<0$:
then equation (3) becomes :

$$Q^{2\beta}(t)+\mu^2Q(t)=0.$$

Then the auxiliary equation of equation (3) becomes $r^2+\mu^2=0$, we get two distinct complex solution $r_1=i\mu$, $r_2=-i\mu$, and since the real part in r_1 and r_2 equal 0.
Therefore the solution is given as follows :

$$Q(t)=c_1\sin(\mu\frac{t^\beta}{\beta})+c_2\cos(\mu\frac{t^\beta}{\beta}).$$

for some constants c_1 and c_2.
Using conditions to get :

$$Q(0) = c_2 = 0; \quad Q(1) = c_1 \sin\frac{\mu}{\beta} = 0.$$

Thus

$$\sin\frac{\mu}{\beta} = 0,\ so\ \frac{\mu}{\beta} = n\pi,\ ie\ \mu = n\pi\beta$$

where $n = 1, 2, 3, ...$
Hence

$$Q_n(t) = c_n\ \sin(n\pi t^\beta). \quad (5)$$

Now, we go back to equation (4) we find :

$$-n^2\pi^2\beta^2 P^{2\alpha}(x) - P^\alpha(x) + n^2\pi^2\beta^2 P(x) = 0. \quad (6)$$

The auxiliary equation of equation (6) is :

$$-n^2\pi^2\beta^2 r^2 - r + n^2\pi^2\beta^2 = 0.$$

Thus

$$\Delta = 1 + 4n^4\pi^4\beta^4 > 0.$$

So

$$r_1 = \frac{-1+\sqrt{\Delta}}{2n^2\pi^2\beta^2}\ \ and\ \ r_2 = \frac{-1-\sqrt{\Delta}}{2n^2\pi^2\beta^2}.$$

Hence the solution of this equation is :

$$P(x) = k_1 e^{r_1\frac{x^\alpha}{\alpha}} + k_2 e^{r_2\frac{x^\alpha}{\alpha}}.$$

for some constants k_1 and k_2.
Conditions implies that :

$$P(0) = k_1 + k_2 = 0,\ \ so\ \ k_2 = -k_1,\ \ thus$$
$$P(x) = k_1(e^{r_1\frac{x^\alpha}{\alpha}} - e^{r_2\frac{x^\alpha}{\alpha}}).$$

Hence

$$P_n(x) = k_n(e^{r_1\frac{x^\alpha}{\alpha}} - e^{r_2\frac{x^\alpha}{\alpha}}). \quad (7)$$

Combining (5) and (7) to get :

$$U(x,t) = \sum_{n=1}^{\infty} b_n(e^{r_1\frac{x^\alpha}{\alpha}} - e^{r_2\frac{x^\alpha}{\alpha}})\sin(n\pi t^\beta).$$

Conditions show that :

$$t^{2\beta} = \sum_{n=1}^{\infty} b_n(r_1 e^{r_1\frac{1}{\alpha}} - r_2 e^{r_2\frac{1}{\alpha}})\sin(n\pi t^\beta).$$

Using the $\beta-$Fractional Fourier series of $t^{2\beta}$, we find that :

$$\begin{aligned} b_n &= \frac{2\beta}{p^\beta}\int_0^p t^{2\beta}\sin(n\pi t^\beta)\frac{dt}{t^{1-\beta}} \\ &= \frac{2\beta}{p^\beta}\int_0^p t^{3\beta-1}\sin(n\pi t^\beta)dt \end{aligned}$$

Now we have to use this variable change $u = t^\beta$ and $du = \beta t^{\beta-1}$ to get :

$$\begin{aligned} b_n &= \frac{2\beta}{p^\beta}\int_0^{p^\beta} t^{3\beta-1}\sin(n\pi u)\frac{t^{1-\beta}}{\beta}du \\ &= \frac{2}{p^\beta}\int_0^{p^\beta} u^2\sin(n\pi u)du. \end{aligned}$$

Then :

$$b_n = \frac{2}{p^\beta}\left[-u^2\frac{\cos(n\pi u)}{n\pi} + 2u\frac{\sin(n\pi u)}{n^2\pi^2} + 2\frac{\cos(n\pi u)}{n^3\pi^3}\right]_0^{p^\beta}.$$

Thus :

$$b_n = -2p^\beta\frac{cos(n\pi p^\beta)}{n\pi} + 4\frac{\sin(n\pi p^\beta)}{n^2\pi^2} + 4\frac{\cos(n\pi p^\beta)}{p^\beta n^3\pi^3} - \frac{4}{p^\beta n^3\pi^3}.$$

Therefore :

$$U(x,t) = \sum_{n=1}^{\infty} b_n(e^{r_1\frac{x^\alpha}{\alpha}} - e^{r_2\frac{x^\alpha}{\alpha}})\sin(n\pi\beta\frac{t^\beta}{\beta}).$$

Where :

$$b_n = -2p^\beta\frac{\cos(n\pi p^\beta)}{n\pi} + 4\frac{\sin(n\pi p^\beta)}{n^2\pi^2} + 4\frac{cos(n\pi p^\beta)}{p^\beta n^3\pi^3} - \frac{4}{p^\beta n^3\pi^3}.$$

5 Application

Now, as an application we will use fractional Fourier series to solve the Conformable Fractional heat partial differential equation as demonstrated in the example below.

Example 14

$$\begin{aligned} \frac{\partial^\alpha}{\partial t^\alpha}\left(\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}\right) &= c^2\frac{\partial^2 u(x,t)}{\partial x^2} && (1) \\ 0 < t \quad &, \quad 0 < x < L \\ u(0,t) &= 0 && (2) \\ u(L,t) &= 0 && (3) \\ \frac{\partial u}{\partial x}(0,t) &= t^{2\alpha} && (4) \\ u(x,0) &= 0 && (5) \end{aligned}$$

Solution:
Let us use separation of variables technique.
So let $u(x,t) = P(x)Q(t)$, substitute in the equation to get

$$P(x)Q^{2\alpha}(t) = c^2 P^2(x)Q(t).$$

Since x and t are independent variables, then we get

$$\frac{Q^{2\alpha}(t)}{Q(t)} = \frac{c^2P^2(x)}{P(x)} = \lambda.$$

for some constant λ.
Consequently,

$$Q^{2\alpha}(t) - \lambda Q(t) = 0 \tag{6}$$
$$c^2 P^2(x) - \lambda P(x) = 0 \tag{7}$$

We start with equation (7), $P^2(x)-\lambda P(x) = 0$. *Then the auxiliary equation of equation* (7) *is* $r^2 - \frac{\lambda}{c^2} = 0$. *Now there are three possibilities for* λ *:*

Case (i) : $\lambda = 0$. *Then* $r = 0$, *so the solution is* $P(x) = c_1 + c_2 x$.
Using condition (2), *we get* $c_1 = 0$, *again using condition* (3), *we get* $c_2 = 0$.
So $P(x) = 0$. *Hence, no non trivial solution when* $\lambda = 0$.

Case (ii) : $\lambda = \mu^2 > 0$. *So* $r = +\frac{\mu}{\lambda}$, $r = -\frac{\mu}{\lambda}$, *then the solution is* $P(x) = c_1 \sinh \frac{\mu}{c} x + c_2 \cosh \frac{\mu}{c} x$.
Using condition (2) *and* (3), *we get* $c_1 = 0$ *and* $c_2 = 0$, *so* $P(x) = 0$.
Finally, no non trivial solution when $\lambda > 0$.

Case (iii) : $\lambda = -\mu^2 < 0$. *so* $r = +i\frac{\mu}{\lambda}$, $r = -i\frac{\mu}{\lambda}$, *then the solution is* $P(x) = c_1 \sin \frac{\mu}{c} x + c_2 \cos \frac{\mu}{c} x$.
Using condition (2), *we get* $c_2 = 0$, *also condition* (3), *we get*
$P(L) = c_1 \sin \frac{\mu}{c} L = 0$, *so* $\sin \frac{\mu}{c} L = 0$.
Hence, $\mu = \frac{c}{L} n\pi$, *n = 1, 2, 3, ... which gives that* $\lambda = -(\frac{c}{L} n\pi)^2$, *n= 1,2,3,...*
Thus

$$P_n(x) = c_n \sin(\frac{n\pi}{L} x) \tag{8}$$

Now, we return back to equation (6),

$$Q^{2\alpha}(t) - \lambda Q(t) = 0.$$

So

$$Q^{2\alpha}(t) + (\frac{c}{L} n\pi)^2 Q(t) = 0.$$

Its auxiliary equation is $r^2 + (\frac{c}{L} n\pi)^2 = 0$, *thus we get two distinct complex solution*

$$r_1 = i\frac{c}{L} n\pi,\ r_2 = -i\frac{c}{L} n\pi$$

and since the real part equal 0, the solution is given as follow:

$$Q(t) = b_1\ \cos(\frac{c}{L} n\pi \frac{t^\alpha}{\alpha}) + b_2\ \sin(\frac{c}{L} n\pi \frac{t^\alpha}{\alpha}).$$

Condition (5) *implies that* $b_1 = 0$.
So

$$Q(t) = b_2\ \sin(\mu \frac{t^\alpha}{\alpha}).$$

Hence,

$$Q_n(t) = b_n\ \sin(n\pi \frac{c}{L} \frac{t^\alpha}{\alpha}) \tag{9}$$

Combining (8) *and* (9) *to get*

$$u(x,t) = \sum_{n=1}^{\infty} a_n\ \sin(\frac{n\pi}{L} x)\ \sin(n\pi \frac{c}{L} \frac{t^\alpha}{\alpha}).$$

Now, using condition (4), $\frac{\partial u}{\partial x}(0,t) = t^{2\alpha}$, *to get*

$$t^{2\alpha} = \sum_{n=1}^{\infty} a_n \frac{n\pi}{L}\ \sin(n\pi \frac{c}{L} \frac{t^\alpha}{\alpha}).$$

Using the $\alpha-$*Fractional Fourier series of* $t^{2\alpha}$, *we find that*

$$\begin{aligned} a_n &= \frac{2\alpha}{p^\alpha} \int_0^p t^{2\alpha} \sin(n\frac{t^\alpha}{\alpha}) \frac{dt}{t^{1-\alpha}} \\ &= \frac{2\alpha}{p^\alpha} \int_0^p t^{3\alpha-1} \sin(n\frac{t^\alpha}{\alpha}) dt. \end{aligned}$$

By using the substitution $u = t^\alpha$, *and* $du = \alpha t^{\alpha-1} dt$, *then*

$$\begin{aligned} a_n &= \frac{2\alpha}{p^\alpha} \int_0^{p^\alpha} t^{3\alpha-1} \sin(n\frac{u}{\alpha}) \frac{t^{1-\alpha}}{\alpha} du \\ &= \frac{2}{p^\alpha} \int_0^{p^\alpha} t^{2\alpha} \sin(n\frac{u}{\alpha}) du \\ &= \frac{2}{p^\alpha} \int_0^{p^\alpha} u^2 \sin(n\frac{u}{\alpha}) du. \end{aligned}$$

Consequently

$$\begin{aligned} a_n &= \frac{2}{p^\alpha} \left[-u^2 \cos(\frac{n}{\alpha} u).\frac{\alpha}{n} + 2u\ \sin(\frac{n}{\alpha} u).\frac{\alpha^2}{n^2} \right]_0^{p^\alpha} \\ &+ \frac{2}{p^\alpha} \left[2\cos(\frac{n}{\alpha} u).\frac{\alpha^3}{n^3} \right]_0^{p^\alpha} \\ &= \frac{2}{p^\alpha} \left[-p^{2\alpha} \cos(\frac{n}{\alpha} p^\alpha).\frac{\alpha}{n} + 2p^\alpha\ \sin(\frac{n}{\alpha} p^\alpha).\frac{\alpha^2}{n^2} \right] \\ &+ \frac{2}{p^\alpha} \left[2\cos(\frac{n}{\alpha} p^\alpha).\frac{\alpha^3}{n^3} \right] - 2\frac{\alpha^3}{n^3} \\ &= -2p^\alpha \cos(\frac{n}{\alpha} p^\alpha).\frac{\alpha}{n} + 4\sin(\frac{n}{\alpha} p^\alpha).\frac{\alpha^2}{n^2} \\ &+ \frac{4}{p^\alpha} \cos(\frac{n}{\alpha} p^\alpha).\frac{\alpha^3}{n^3} - 2\frac{\alpha^3}{n^3}. \end{aligned}$$

Therefore

$$u(x,t) = \sum_{n=1}^{\infty} a_n\ \sin(\frac{n\pi}{L} x)\ \sin(n\pi \frac{c}{L} \frac{t^\alpha}{\alpha}).$$

Where

$$\begin{aligned} a_n &= -2p^\alpha \cos(\frac{n}{\alpha}p^\alpha).\frac{\alpha}{n} + 4\sin(\frac{n}{\alpha}p^\alpha).\frac{\alpha^2}{n^2} \\ &+ \frac{4}{p^\alpha}\cos(\frac{n}{\alpha}p^\alpha).\frac{\alpha^3}{n^3} - 2\frac{\alpha^3}{n^3}. \end{aligned}$$

IN our paper, we use the mathematica coding programm to see the $\alpha-$Fractional Fourier series approximation and the classical Fourier series approximation of functions.

The figures bellow represent the solution of Conformable Fractional heat partial differential equation with three different terms.

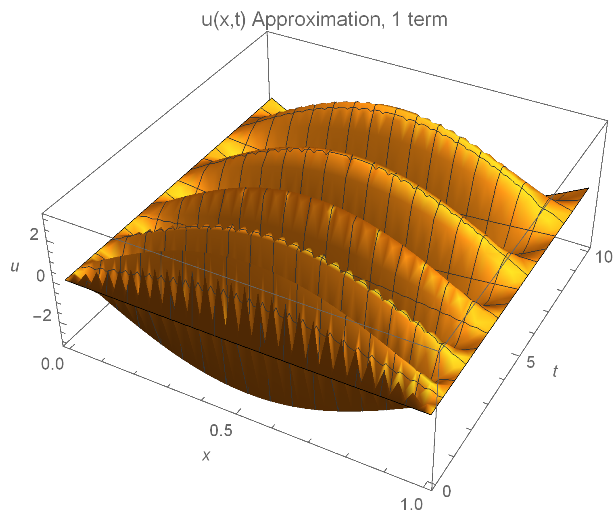

Figure 10: u(x,t) Approximation, 1 term.

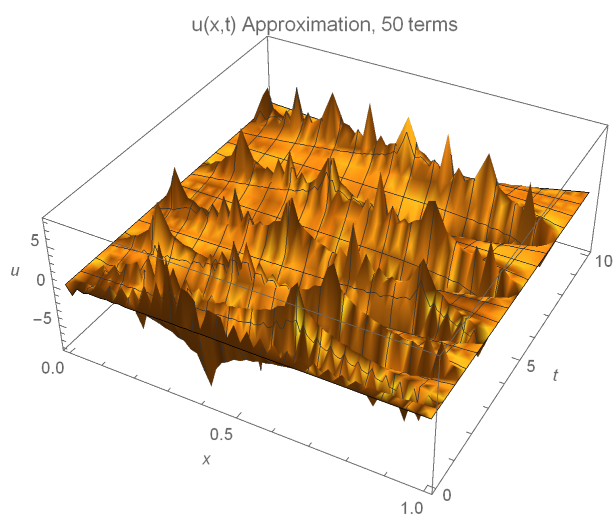

Figure 11: u(x,t) Approximation, 50 terms.

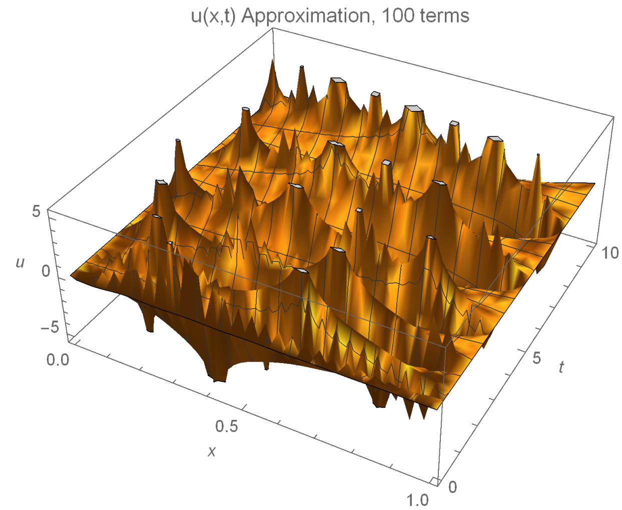

Figure 12: u(x,t) Approximation, 100 terms.

6 Conclusion

The results explained in the previous sections show that the separation variables with fractional Fourier series technique solves some difficult problems that cannot be solved with classical methods, also we can use the conformable fractional Fourier series approximation to approach some functions.

Closing remark:
(i) In the Example 11, $\alpha = \frac{1}{2}$ give best approximation of the function $f(t)$ then $\alpha = \frac{1}{4}$.
(ii) Is the $\alpha-$Fractional Fourier series give best approximation of functions then the classical Fourier series and vice versa ?
(iii) What is the value of α which give the best and perfect approximation in $\alpha-$Fractional Fourier series approximation ? and how to fix it ?

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