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The Structure of Higher Degree Symmetry Classes of Tensors*

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The paper is concerned with symmetry classes of tensors which arise from a permutation group G and irreducible character χ of G. In case χ is of degree 1, a well-known algorithm is available for inducing a basis of the symmetry class from the underlying vector space. When the degree of χ is greater than 1, no comparable construction has been discovered. The difficulties are discussed and results obtained in some special cases.

Key words: Decomposable (or pure) tensor products; irreducible complex character; orthogonality relations; permutation group.

1. Introduction

Let V be a complex inner product space of dimension n. Let $\overset{m}{\otimes} V$ denote the mth tensor power of V, and let $v_1 \otimes \ldots \otimes v_m$ be the (pure or decomposable) tensor product of the indicated vectors. The inner product in V induces an inner product in $\overset{m}{\otimes} V$ which is completely determined by its action on the set of decomposable tensors, namely

$$(v_1 \otimes \ldots \otimes v_m, w_1 \otimes \ldots \otimes w_m) = \prod_{t=1}^{m} (v_t, w_t). \tag{1}$$

By S_m, we mean the full symmetric permutation group on $\{1, \ldots, m\}$. If $\sigma \in S_m$, there is a (unique) linear operator $P(\sigma^{-1})$ on $\overset{m}{\otimes} V$ which has the effect $P(\sigma^{-1}) v_1 \otimes \ldots \otimes v_m = v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(m)}$, for all $v_1, \ldots, v_m \in V$. It follows that $P(\sigma)\, P(\pi) = P(\sigma\pi)$. Moreover, from (1), $P(\sigma)^* = P(\sigma^{-1})$. Let G be a subgroup of S_m, and χ an irreducible (complex) character of G. Define

$$T(G, \chi) = \frac{\chi(id)}{o(G)} \sum_{\sigma \in G} \chi(\sigma)\, P(\sigma),$$

where id = identity of G, and $o(G)$ is the order of G. By the orthogonality relations for characters, $T(G, \chi)$ is an orthogonal projection onto its range $V_\chi(G)$ (see, e.g., [5][1] or [12]). The subspace $V_\chi(G)$ is called a symmetry class of tensors [8]. Several authors have exploited these symmetry classes to obtain information about so called generalized matrix functions (see, e.g., [5], [8], [9], and [11]).

Until recently, however, most of the work has involved only linear characters. One reason for this preference is the existence, in the case $\chi(id) = 1$, of a convenient basis for $V_\chi(G)$ which is induced from a given basis of V. In the case $\chi(id) > 1$, it is not easy to obtain such a basis. A more precise idea of our interest must await further introductory material.

With $\Gamma_{m,n}$ we denote the set of functions from the first m positive integers to the first n. It is convenient to think of $\Gamma_{m,n}$ as a set of integer sequences of length m. Thus

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$$\Gamma_{m,n} = \{\gamma = (\gamma(1), \ldots, \gamma(m)): \quad 1 \leq \gamma(t) \leq n, \qquad 1 \leq t \leq m\}.$$

If $e_1, \ldots, e_n$ is an orthonormal basis of V, it is well known (see, e.g., [8]) that $\{e^{\otimes}_{\gamma} = e_{\gamma(1)} \otimes \ldots \otimes e_{\gamma(m)} : \gamma \in \Gamma_{m,m}\}$ is an o.n. basis of $\overset{m}{\otimes} V$. It follows that $\{e^*_{\gamma} = T(G, \chi) e^{\otimes}_{\gamma} : \gamma \in \Gamma_{m,n}\}$ must span $V_{\chi}(G)$. (In general, write $x_1 * \ldots * v_m = T(G, \chi) v_1 \otimes \ldots \otimes v_m$.) If $\alpha, \beta \in \Gamma_{m,n}$, observe that

$$\begin{aligned}(e^*_{\alpha}, e^*_{\beta}) &= (T(G,\chi)e^{\otimes}_{\alpha}, T(G,\chi)e^{\otimes}_{\beta}) \\ &= (T(G,\chi)e^{\otimes}_{\alpha}, e^{\otimes}_{\beta}) \\ &= \frac{\chi(id)}{o(G)} \sum_{\sigma \in G} \chi(\sigma^{-1}) \prod_{t=1}^{m} (e_{\alpha\sigma(t)}, e_{\beta(t)}).\end{aligned} \tag{2}$$

It follows from (2) that $(e^*_{\alpha}, e^*_{\beta}) = 0$ unless there is a $\pi \in G$ such that $\beta = \alpha\pi$. We will say that $\alpha \equiv \beta$ (mod G) if there exists a $\pi \in G$ such that $\beta = \alpha\pi$. Clearly, "$\equiv$ (mod G)" is an equivalence relation.

If $\beta = \alpha\pi$, for some fixed $\pi \in G$, then

$$\begin{aligned}(e^*_{\alpha}, e^*_{\beta}) &= \frac{\chi(id)}{o(G)} \sum_{\sigma \in G} \chi(\sigma) \prod_{t=1}^{m} (e_{\alpha(t)}, e_{\alpha\pi\sigma(t)}) \\ &= \frac{\chi(id)}{o(G)} \sum_{\tau \in G} \chi(\pi^{-1}\tau) \prod_{t=1}^{m} (e_{\alpha(t)}, e_{\alpha\tau(t)}) \\ &= \frac{\chi(id)}{o(G)} \sum_{\tau \in G_{\alpha}} \chi(\pi^{-1}\tau),\end{aligned} \tag{3}$$

where $G_{\alpha} = \{\tau \in G: \quad \alpha\tau = \alpha\}$ is the stabilizer subgroup of α. In particular, by taking $\pi = id$ in (3) one sees that $e^*_{\alpha} \neq 0$, if and only if

$$\alpha \in \Omega = \{\gamma \in \Gamma_{m,n}: \quad \sum_{\sigma \in G_{\gamma}} \chi(\sigma) \neq 0\},$$

i.e., Ω consists of those sequences γ which have the property that the restriction of χ to G_{γ} contains the identically 1 character as a component. (Although not explicit in the notation, Ω depends on m, n, G and χ.) It follows that $\{e^*_{\omega}: \omega \in \Omega\}$ spans $V_{\chi}(G)$.

Now, if $\alpha \equiv \beta$ (mod G), then G_{α} is conjugate to G_{β}. Therefore, Ω is a union of equivalence classes, i.e., if $\alpha \equiv \beta$(mod G), then $e^*_{\alpha} = 0$ if and only if $e^*_{\beta} = 0$. Let $\bar{\Delta}$ be a system of distinct representatives for the equivalence classes in Ω. (In practice, $\bar{\Delta}$ is usually chosen to consist of those elements of Ω which come first, in lexicographic order, in their equivalence classes.) Then

$$\Omega = \bigcup_{\alpha \in \bar{\Delta}} \{\alpha\sigma: \quad \sigma \in G\}. \tag{4}$$

THEOREM A ([10]): *Let* $e_1, \ldots, e_n$ *be a basis of* V. *Then* $V_{\chi}(G)$ *is the direct sum of the spaces* $\langle e^*_{\alpha\sigma}: \quad \sigma \in G\rangle$, *as* α *ranges over* $\bar{\Delta}$. (*The angular brackets denote linear closure.*)

PROOF: Choose the inner product on V with respect to which $e_1, \ldots, e_n$ is orthonormal. The theorem follows from (4) and the definitions.

The result which makes the degree one case so fruitful is this:

THEOREM B (Marcus and Minc [9]): *Let* $e_1, \ldots, e_n$ *be a basis of* V. *Suppose* $\chi(\text{id}) = 1$. *Then* $\{e^*_{\alpha}: \quad \alpha \in \bar{\Delta}\}$ *is a basis of* $V_{\chi}(G)$.

PROOF: It is routine to verify that $P(\sigma)$ commutes with $T(G, \chi)$ for all $\sigma \in G$. Moreover, if $\chi(id) = 1$, then $P(\sigma)T(G, \chi) = \chi(\sigma^{-1})T(G, \chi)$. It follows that $e^*_{\alpha\sigma} = \chi(\sigma)e^*_\alpha$ if $\chi(id) = 1$. So, each subspace in the direct sum of Theorem A is one dimensional.

That $\{e^*_\alpha: \alpha \in \bar{\Delta}\}$ is not a basis of $V_\chi(G)$ when $\chi(id) > 1$ is evident from the following result of S. Pierce [12]:

THEOREM C: *Let $\alpha \in \bar{\Delta}$ be arbitrary. There is a $\sigma \in G$ such that e^*_α and $e^*_{\alpha\sigma}$ are linearly independent if and only if $\chi(\mathrm{id}) > 1$.*

R. Freese [5] has improved Theorem C. Let $s_\alpha = \dim\langle e^*_{\alpha\sigma}: \sigma \in G\rangle$. Freese's result is this:

THEOREM D. *If $\alpha \in \Gamma_{m,n}$, then*

$$s_\alpha = \chi(\mathrm{id})\, (\chi, 1)_{G_\alpha},$$

i.e., s_α is $\chi(\mathrm{id})$ times the number of occurrences of the identically one character in the restriction of χ to G_α.

To conclude this section, we list a number of facts about s_α which follow from our discussion above.

(i) $s_\alpha \neq 0$, if and only if $\alpha \in \Omega$.

(ii) $\sum_{\alpha \in \bar{\Delta}} s_\alpha = \dim V_\chi(G)$

(iii) $\chi(id) \leq s_\alpha < \chi(id)^2$, for all $\alpha \in \Omega$.

(iv) $s_\alpha \leq [G:G_\alpha]$, for all α. (In fact, it is clear from Freese's proof of Theorem D that $s_\alpha < [G:G_\alpha]$ unless χ is identically 1 and $G = G_\alpha$.)

(v) $\|e^*_\alpha\|^2 = s_\alpha/[G:G_\alpha]$, if $e_1, \ldots, e_n$ is an o.*n*. basis of V. (See (3).)

2. Results

Presently, the outstanding problem is to choose from $\{e^*_{\alpha\sigma}: \sigma \in G\}$ a basis of $\langle e^*_{\alpha\sigma}: \sigma \in G\rangle$. In this generality, the task seems quite difficult. We are able to supply an answer (Theorem 4 below) only in a very special situation.

As a first step toward analyzing the dependence relations among the elements of $\{e^*_{\alpha\sigma} : \sigma \in G\}$, $\alpha \in \Omega$, one is naturally led to consider

$$G^\alpha = \{\sigma \in G: \text{ there exists } c_\alpha(\sigma) \text{ such that } e^*_{\alpha\sigma} = c_\alpha(\sigma)e^*_\alpha\}.$$

(If $\chi(id) = 1$, then $G^\alpha = G$ and $c_\alpha = \chi$. Moreover, $G_\alpha \subseteq G^\alpha$ for all $\alpha \in \Omega$.)

We first claim that G^α does not depend on the basis $e_1, \ldots, e_n$. Let $v_1, \ldots, v_n$ be another basis of V. Define a linear operator T on V by $T(e_i) = v_i$, $1 \leq i \leq n$, and linear extension. It is well known (see, e.g., [10]) that T induces a linear operator $K(T)$ on $V_\chi(G)$ such that

$$K(T)(x_1 * \ldots * x_m) = (Tx_1) * \ldots * (Tx_m),$$

for all $x_1, \ldots, x_m \in V$. Since T is invertible, it follows that $K(T)$ is invertible. Indeed, $K(T)^{-1} = K(T^{-1})$. Applying $K(T)$ to both sides of the equation $e^*_{\alpha\sigma} = c_\alpha(\sigma)\, e^*_\alpha$ one obtains $v^*_{\alpha\sigma} = c_\alpha(\sigma)\, v^*_\alpha$.

THEOREM 1: *For all $\alpha \in \Omega$, G^α is a group and c_α is a linear character on it.*

PROOF: If $\sigma, \pi \in G$, then

$$\begin{aligned} e^*_{\alpha\sigma\pi} &= P(\pi^{-1})\, e^*_{\alpha\sigma} \\ &= c_\alpha(\sigma)\, P(\pi^{-1})\, e^*_\alpha \\ &= c_\alpha(\sigma)\, e^*_{\alpha\pi} \\ &= c_\alpha(\sigma)\, c_\alpha(\pi)\, e^*_\alpha. \end{aligned}$$

Thus $\sigma\pi \in G^\alpha$ and, since $e^*_\alpha \neq 0, c_\alpha(\sigma\pi) = c_\alpha(\sigma)\, c_\alpha(\pi)$.

We remark (without proof since the result seems peripheral to the present undertaking) that the restriction of χ to G^α contains c_α as a component for all $\alpha \in \Omega$, i.e.,

$$(\chi, 1)_{G_\alpha} > 0 \text{ implies } (\chi, c_\alpha)_{G^\alpha} > 0.$$

The converse fails.

COROLLARY 1: *If* $\alpha \in \Omega$, *then* $s_\alpha \leq [G: \; G^\alpha]$. (*Indeed, if* S^α *is a system of right coset representatives for* G^α *in* G, *then* $\bigcup_{\alpha \in \bar{\Delta}} \{e^*_{\alpha\pi}: \; \pi \in S^\alpha\}$ *spans* $V_\chi(G)$.)

(It follows from Corollary 1 and (iii) of section 1 that $\chi(id) \leq [G: \; G^\alpha]$ for all $\alpha \in \Omega$. This inequality may be of some interest in itself because G^α is generally not normal in G [1, Theorem (53.17)].)

EXAMPLE 1: Let $G = S_3$. Let χ be the irreducible character of G of degree 2, and take $\alpha = (1, 1, 2)$. Then $G_\alpha = S_2$. If G_α were not all of G^α, then G^α would be all of S_3, implying that $[G: \; G^\alpha] = 1 < s_\alpha = 2$, contradicting Corollary 1. Therefore, $G_\alpha = G^\alpha$, and $[G: \; G^\alpha] = 3$. In particular, it's not true in general that $s_\alpha = [G: \; G^\alpha]$.

Subsequent developments will make clearer the relationship between G_α and G^α. We now make another definition. Let G be a subgroup of S_m. Let χ be an irreducible character of G. Define

$$G_\chi = \{\sigma \in G: \; |\chi(\sigma)| = \chi(id)\}.$$

It is easy to see that G_χ is a normal subgroup of G and $\lambda = \chi/\chi(id)$ is a linear character on it [4, p. 35], [11]. In fact, G_χ consists of those σ which are represented by scalars in any representation which affords χ.

THEOREM 2: *For* $\alpha \in \Omega$, $G_\chi \subseteq G^\alpha$, *and the restriction of* c_α to G_χ is λ.

PROOF: Let $\sigma \in G_\chi$. Then

$$\begin{aligned} e^*_{\alpha\sigma} &= \frac{\chi(id)}{o(G)} \sum_{\pi \in G} \chi(\pi) P(\pi\sigma^{-1})\, e_{\bar{\alpha}} \\ &= \frac{\chi(id)}{o(G)} \sum_{\pi \in G} \chi(\pi\sigma) P(\pi)\, e_{\bar{\alpha}} \\ &= \frac{\chi(id)}{o(G)} \lambda(\sigma) \sum_{\pi \in G} \chi(\pi) P(\pi)\, e_{\bar{\alpha}} \\ &= \lambda(\sigma)\, e^*_\alpha. \end{aligned}$$

COROLLARY 2: *For all* $\alpha \in \Omega$, $G_\alpha G_\chi \subseteq G^\alpha$.

EXAMPLE 2: It is tempting to conjecture that $G_\alpha G_\chi = G^\alpha$. Unfortunately, this is not always the case. Let $G = S_5$. Suppose χ arises from the frame (3, 2). Let $\alpha = (1,1,1,2,3)$ and let σ be the transposition (45). Then $\alpha\sigma = (1,1,1,3,2)$, and a brute force computation shows that $e^*_\alpha = e^*_{\alpha\sigma}$. In particular, since $G\chi = \{id\}$ and $G_\alpha = S_3$, it follows that $(45) \in G^\alpha \backslash G_\alpha G_\chi$.

It was proved in [11] that $\chi(id)^2 \leq [G: \; G_\chi]$, so the inequality $s_\alpha \leq [G: \; G_\chi]$ which arises from Theorem 2 is not very interesting. However, one might be tempted to conjecture that $\chi(id)^2 \leq [G: \; G_\alpha G_\chi]$ for all $\alpha \in \Omega$. A counterexample follows.

EXAMPLE 3. Let G be the subgroup of S_4 generated by $\{(14)(23), (1234)\}$. Then G is the dihedral group D_4 of order 8. Let χ be the irreducible character of G of degree 2. Then $\chi(id) = 2 = -\chi((13)(24))$, and χ is zero on the rest of G. Thus, $G_\chi = \{id, \; (13)(24)\}$. If $\alpha = (1,1,2,2)$, then $G_\alpha = \{id, \; (12)(34)\}$, and $\alpha \in \bar{\Delta}$. Moreover, $G_\alpha G_\chi = \{id, \; (12)(34), \; (13)(24), \; (14)(23)\}$ and $[G: \; G_\alpha G_\chi] = 2$, which is less than $\chi(id)^2 = 4$.

It is worth pointing out some other features of Example 3: Since

$$1 < s_\alpha \leq [G: \; G^\alpha] \leq [G: \; G_\alpha G_\chi] = 2,$$

it follows that $s_\alpha = [G: \; G_\alpha G_\chi]$, and hence $G^\alpha = G_\alpha G_\chi$. Moreover, $\chi(id)^2 = [G: \; G_\chi]$. In a moment, we shall see that these observations are connected. First, however, it should be mentioned that the case of equality in $\chi(id)^2 \leq [G: \; G_\chi]$ is related to some recent work of F. DeMeyer, S. M. Gagola, G. Janusz, K. M. Timmer, and J. Yellen ([2], [3], [6], [13], and [14]) in which the case of equality in $\chi(id)^2 \leq [G: \; Z(G)]$ is studied. In particular, since $Z(G) \subseteq G_\chi$, $[G: \; Z(G)] = \chi(id)^2$ implies $[G: \; G_\chi] = \chi(id)^2$.

THEOREM 3: *If* $\chi(id)^2 = [G: \; G_\chi]$, *then* $s_\alpha = [G: \; G_\alpha G_\chi]$ *(and therefore* $G^\alpha = G_\alpha G_\chi$*) for all* $\alpha \in \Omega$.

PROOF: If $[G: \; G_\chi] = \chi(id)^2$, then $\chi(\sigma) \neq 0$, if and only if $\sigma \in G_\chi$ [11]. It follows from Theorem D that

$$s_\alpha = \frac{\chi(id)}{o(G_\alpha)} \sum_{\sigma \in G_\alpha} \chi(\sigma)$$

$$= \frac{\chi(id)^2}{o(G_\alpha)} \sum_{\sigma \in G_\alpha \cap G_\chi} \lambda(\sigma),$$

where $\lambda = \chi/\chi(id)$. Since $s_\alpha \neq 0$, and since λ is a linear character, it must be that $\lambda(\sigma) = 1$ for all $\sigma \in G_\alpha \cap G_\chi$. Thus

$$s_\alpha = \chi(id)^2 \, o(G_\alpha \cap G_\chi)/o(G_\alpha)$$

$$= o(G) \, o(G_\alpha \cap G_\chi)/o(G_\chi) \, o(G_\alpha)$$

$$= [G: \; G_\alpha G_\chi]$$

from elementary group theory (see, e.g., [7, p. 45]).

THEOREM 4: *Let* $e_1, \ldots, e_n$ *be an o.n. basis of* V. *Let* $\pi_1, \ldots, \pi_k$ *be right coset representatives for* G_χ *in* G. *Suppose* $\chi(id)^2 = [G: \; G_\chi]$. *If* $\alpha \in \Omega$ *is such that* $G_\alpha \subseteq G_\chi$, *then* $\{e^*_{\alpha\pi_i}: 1 \leq i \leq [G: \; G_\chi]\}$ *is an orthogonal basis of* $\langle e^*_{\alpha\sigma} : \sigma \in G\rangle$.

PROOF. Let $\lambda(\sigma) = \chi(\sigma)/\chi(id)$, $\sigma \in G_\chi$. Since $\alpha \in \Omega$, it follows that λ is identically 1 on G_α.

Now,

$$\left(e^*_{\alpha\pi_i}, e^*_{\alpha\pi_j}\right) = \left(e^*_{\alpha}, e^*_{\alpha\pi_j\pi_i-1}\right)$$

$$= \frac{\chi(id)}{o(G)} \sum_{\tau \in G_\alpha} \chi\left(\pi_i \pi_j{}^{-1}\tau\right), \text{ from (3)}$$

$$= \frac{\chi(id)}{o(G)} \chi\left(\pi_i\pi_j{}^{-1}\right) \sum_{\tau \in G_\alpha} \lambda(\tau)$$

$$= \chi(id)\ \chi\left(\pi_i\pi_j{}^{-1}\right)/[G:\quad G_\alpha].$$

The result follows because $\chi(\pi_i\pi_j{}^{-1}) \neq 0$ if and only if $i = j$ (again appealing to [11]).

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